# Learning Consensus via Majority Vote on Dynamic Community Networks

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## Abstract

We introduce two models of consensus following a majority rule on time-evolving stochastic block models (SBM), in which the network evolution is Markovian or non-Markovian. Under the majority rule, in each round, each agent simultaneously updates their opinion according to the majority of their neighbors. Our network has a community structure and randomly evolves with time. In contrast to the classic setting, the dynamics is not purely deterministic, and reflects the structure of SBM by resampling the connections at each step, making agents with the same opinion more likely to connect than those with different opinions.

In the *Markovian model*, connections between agents are resampled at each step according to the SBM law and each agent updates their opinion via the majority rule. We prove a *power-of-one* type result, i.e., any initial bias leads to a non-trivial advantage of winning in the end, uniformly in the size of the network.

In the *non-Markovian model*, a connection between two agents is resampled according to the SBM law only when some of the two changes opinion and is otherwise kept the same. We study the phase transition between the fast convergence to the consensus and a halt of the dynamics. Moreover, we establish thresholds of the initial lead for various convergence speeds.

## 1   Introduction

In the theory of social learning, consensus refers to the following problem: given a collection of agents holding different opinions, the agents interact and update their opinions under certain rules with the goal of reaching unanimity. One of the most classic and straightforward rules for updating opinions is the *majority dynamics*, where in each round, all agents simultaneously update their opinions based on the majority of their neighbors. Typically, the connections between agents are modeled mathematically using graphs/networks, where vertices represent agents and edges represent connections. Majority dynamics has a long history and allows numerous applications, including economics [EF93, BG98], psychology [CH56], biophysics [MP43], and social choice theory [Gra78, NXX+20]. See also [MT17] for a more recent and detailed account.

Recently there has been surging interest in majority dynamics on random networks. Among various models of majority dynamics, two classes of formulations are particularly popular and technically tractable. We briefly describe their settings as follows.

(a) *Majority dynamics on a static random graph.* Consider the Erdős-Rényi graph $\mathsf{G}(n, p)$ representing the connections of the agents, which is fixed throughout the dynamics. Typically the initial opinions held by the agents are assumed to be biased (meaning that one opinion is held

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by more people than any of the others), either in a deterministic way [TV20, SS22, BD22] or randomly [BCO+16, FKM20, Zeh20, CKLT21]. In the dense regime, the recent breakthrough of [SS22] proved a *power-of-one* result, namely any initial bias leads to a nontrivial advantage of winning in the end, uniformly in the size of the network. In the sparser case, [CKLT21] confirms the same result with random initial opinions. For other random graph models, see e.g., [GZ18] for random regular graphs and [Sha21] for inhomogeneous random graphs.

(b) *k-majority dynamics.* Consider a (possibly random) graph $G$ and a fixed integer $k$. In each round, each agent randomly samples $k$ connections from its neighbors in $G$, with an initial bias on the opinions. In the literature, $G$ may refer to the complete graph [DGM+11, BCN+16, GL18, MMR20], expander graphs (as well as Erdős-Rényi graphs) [CER14, CER+15, CRRS16], and the stochastic block models [CNS19, SS21]. In this context, it was shown that any initial bias leads to a consensus.

A common feature of these models is the symmetry of connections between agents with or without the same opinion. For example, in majority dynamics on $\mathsf{G}(n,p)$, both of the two types of connections are sampled with probability $p$. Nevertheless, there is no reason a priori that an agent draws connections equally likely with those holding the same opinion and those with a different opinion. In this paper, we build majority dynamics models beyond the symmetric setting.

To highlight the community structures of those with the same opinion, we introduce two parameters $p, q \in [0, 1]$ to represent the connecting probabilities, where $p$ is the probability that two agents with the same opinion are connected, and $q$ is the probability that two agents with different opinions are connected. Motivated by real-world scenarios, we will assume $p > q$. This structure is captured by the *stochastic block model* (SBM). Introduced by [HLL83], SBM is a typical model of inhomogeneous random graphs; see [BJR07]. The simplest case of an SBM considers a bipartition of the vertices into two blocks, where edges within a certain block are independently sampled with probability $p$ and otherwise with probability $q$. For more detailed applications in machine learning and computer science, see [Abb17].

We remark that although there is literature concerning $k$-majority dynamics on SBM, these works do not reflect the correspondence between the blocks and different opinions. The SBM appears there only as a generic prototype of the underlying graph of interest.

## 1.1 Models

To be more precise, our models can be mathematically formulated as follows. Throughout this paper, for simplicity, we consider two opinions at presence, denoted by $+$ and $-$. Without loss of generality, we assume an initial bias with $n + \Delta$ opinions $+$ and $n$ opinions $-$, where $\Delta = \Delta(n)$ is a positive integer that may depend on $n$. We are interested in the asymptotic behavior of the model as $n \to \infty$.

**Definition 1** (Majority dynamics)**.** Given a graph $G = (V, E)$ whose vertices are indexed by $[n] := \{1, \ldots, n\}$, each vertex is associated with an initial binary opinion labeled by $W_i = W_i^{(0)} \in \{\pm 1\}$. Consider a bipartition $V = V_+^{(0)} \cup V_-^{(0)}$, where $V_+^{(0)} = \{i \in [n] : W_i^{(0)} = 1\}$ and $V_-^{(0)} = \{i \in [n] : W_i^{(0)} = -1\}$. The *majority dynamics* on $G$ refers to the following process: At every time step, each vertex updates its opinion based on the majority of its neighbors, i.e.,

$$W_i^{(t+1)} = \begin{cases} \operatorname{sign}\left(\displaystyle\sum_{(i,j)\in E} W_j^{(t)}\right), & \text{if } \displaystyle\sum_{(i,j)\in E} W_j^{(t)} \neq 0; \\ W_i^{(t)}, & \text{otherwise.} \end{cases} \tag{1}$$

Let

$$V_+^{(t)} := \left\{ i \in [n] : W_i^{(t)} = +1 \right\}, \quad \text{and} \quad V_-^{(t)} := \left\{ i \in [n] : W_i^{(t)} = -1 \right\}.$$

We say the opinion $+$ or $-$ *wins at day* $t$ if $|V_+^{(t)}| = n$ or $|V_-^{(t)}| = n$, respectively.

**Definition 2** (Stochastic block models). For positive integers $m, n > 0$ and $p, q \in [0, 1]$, the *stochastic block model* $G \sim \mathsf{SBM}(m, n, p, q)$ is constructed as follows. The graph $G$ has $m + n$ vertices, labeled by the elements of $[m + n]$. Each vertex $i \in [m + n]$ has a community label $W_i \in \{-1, +1\}$. The two communities $V_+ := \{i \in [m+n] : W_i = +1\}$ and $V_- := \{i \in [m+n] : W_i = -1\}$ satisfy $|V_+| = m$ and $|V_-| = n$. For distinct $i, j \in [m + n]$, if $W_i W_j = 1$, then the edge $(i, j)$ is in $G$ with probability $p$; otherwise the edge $(i, j)$ is in $G$ with probability $q$.

In classic literature on majority dynamics, the underlying graph is fixed throughout the process, and in our framework, such static models do not reflect the evolution of block structures. For example, if an agent changes their opinion from $-$ to $+$, the marginal distribution of the influence from their neighbors (i.e., edge connections) does not change, which was sampled as if he/she had opinion $-$. Therefore, to maintain the community structure, our graph needs to evolve in time to be consistent with the updates of the opinions. The most natural idea is to resample the edges at each round. Depending on to which extent the edges are resampled, we introduce the following two models.

**Definition 3.** For a graph $G = (V, E)$ whose vertices are labeled by $[n]$ with the binary opinions $W_i \in \{\pm 1\}$, let $\mathbf{W} := (W_1, \ldots, W_n)$ denote the sequence of opinions of each vertex. Given an initialization with $n + \Delta$ vertices with opinion $+$ and $n$ vertices with opinion $-$ on the graph $\mathsf{SBM}(n + \Delta, n, p, q)$, consider the following coupled dynamics $(G_t, \mathbf{W}_t)$ of the vertex opinions and the graph:

   (i) (*Markovian model*). For each day $t \in \mathbb{N}$, after the opinions $\mathbf{W}_t$ are determined based on $(G_{t-1}, \mathbf{W}_{t-1})$, we update the graph $G_t = (V, E_t)$ by resampling the edges between all pairs $(i, j)$ based on the law of $\mathsf{SBM}(|V_+^{(t)}|, |V_-^{(t)}|, p, q)$. The opinions $\mathbf{W}_{t+1}$ are computed via the majority rule (1) on the updated graph $(G_t, \mathbf{W}_t)$.

   (ii) (*Non-Markovian model*). For each day $t \in \mathbb{N}$, after the opinions $\mathbf{W}_t$ are determined based on $(G_{t-1}, \mathbf{W}_{t-1})$, we update the graph $G_t = (V, E_t)$ in the following way. For any pair $(i, j)$, if $W_i^{(t)} = W_i^{(t-1)}$ and $W_j^{(t)} = W_j^{(t-1)}$, then keep the connectivity condition between these nodes, i.e.,

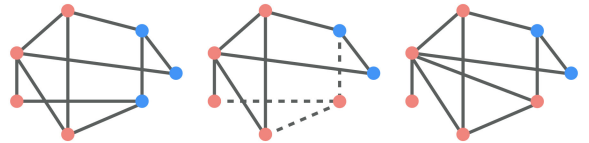$$\mathbf{1}_{\{(i,j) \in E_t\}} = \mathbf{1}_{\{(i,j) \in E_{t-1}\}}.$$

Otherwise, we resample the pair $(i, j)$ based on their updated opinions using the law of $\mathsf{SBM}(|V_+^{(t)}|, |V_-^{(t)}|, p, q)$, i.e.,

$$\mathbb{P}\left[(i, j) \in E_t\right] = \begin{cases} p, & \text{if } W_i^{(t)} W_j^{(t)} = 1; \\ q, & \text{otherwise.} \end{cases}$$

The opinions $\mathbf{W}_{t+1}$ are computed via the majority rule (1) on the updated graph $(G_t, \mathbf{W}_t)$.
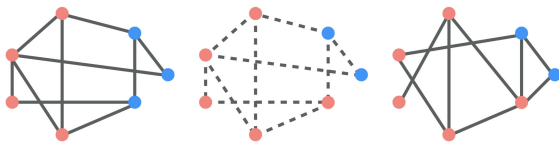


**Figure 1:** The Markovian model: all old connections are resampled



**Figure 2:** The non-Markovian model: only the dashed old connections are resampled

**Remark 1.** In both models, our graph $G_t = (V, E_t)$ is a *dynamic graph*, a graph whose topology evolves with time, in the sense of [CDIG$^+$15]. Dynamic stochastic block models were previously considered in the context of community detection by [CDIG$^+$15]. More generally, Markovian networks have been investigated so far in the context of information spreading [BCP$^+$11] and random walk [CSZ20], etc.

**Remark 2.** For the Markovian model, it is easy to see that the underlying graph is always marginally a stochastic block model, with two blocks given by agents holding the two different opinions. Nevertheless, the marginal distribution of the non-Markovian model is in general *not* simply a stochastic block model due to the dependency of the connections on the prior information of neighbor's opinions, making it more intractable to analyze.

By definition, the evolution of the Markovian model on different days are independent, and hence the number of vertices with opinion $+$, $|V_+^{(t)}|$, $t \in \mathbb{N}_0$, forms a Markov chain on $\{0, \ldots, 2n + \Delta\}$ with absorbing states $\{0, 2n + \Delta\}$. This explains the names of our models.

One motivation for our models is from Social choice theory. In the context of social choice, we may think of our models as an election between two parties. Members of the same party are more united and less likely to be influenced by their opponents. In the non-Markovian model, once a pair of agents interact with each other, their social connection will not break unless one or both sides of the agents change their opinions. Meanwhile, if an agent changes their opinion, their social connections will be rebuilt based on the updated opinion and the current profiles of the two parties. The Markovian model, in contrast, can be regarded as the other extreme case, in which all agents interact with each other randomly at each round. It would be an interesting open question to study an model where the randomness of the resampling is intermediate.

## 1.2 Main Results

Let $G = (V, E) \sim \mathsf{SBM}(n + \Delta, n, p, q)$. Since the cases $p = 0$ or $q = 0$ are trivial, we will always assume $p, q > 0$. Indeed, if $p = 0$, the graph $G$ is bipartite and consequently every agent in the network will alternate their opinion in each step. If $q = 0$, then $G$ is disconnected and separated by the two disjoint communities, meaning that there will not be opinion changes. Throughout the paper, for simplicity we fix two constants $(p, q)$ independent of $n$ with $0 < q < p \leqslant 1$, but the initial bias $\Delta = \Delta(n) \in \mathbb{N}$ may depend on $n$. We remark that our techniques generalize easily to the case $(\log n)^{-c} \leqslant p, q \leqslant 1 - (\log n)^{-c}$ with $p/q \geqslant 1 + \delta$ for some absolute constant $c > 0$ and any $\delta > 0$, as well as the corresponding cases with $q > p$. We now define some events that are of interest.

**Definition 4** (Outcomes of the dynamics). For the coupled majority dynamics $(G_t, \mathbf{W}_t)$ defined in Definition 3, consider the following events.

  (i) The opinion $+$ wins at day $t$,

$$\mathscr{P}_t := \left\{ |V_+^{(t)}| = 2n + \Delta \right\}, \ t \in \mathbb{N}.$$

  (ii) The opinion $+$ wins eventually,

$$\mathscr{P} := \left\{ \lim_{t \to \infty} |V_+^{(t)}| = 2n + \Delta \right\} = \bigcup_{t \in \mathbb{N}} \mathscr{P}_t.$$

  (iii) The opinion $-$ wins at day $t$,

$$\mathscr{M}_t := \left\{ |V_-^{(t)}| = 2n + \Delta \right\}, \ t \in \mathbb{N}.$$

(iv) The opinion $-$ wins eventually,

$$\mathscr{M} := \left\{ \lim_{t \to \infty} |V_-^{(t)}| = 2n + \Delta \right\} = \bigcup_{t \in \mathbb{N}} \mathscr{M}_t.$$

(v) The dynamics halts (before reaching consensus),

$$\mathscr{T} := (\mathscr{P} \cup \mathscr{M})^c.$$

**Remark 3.** Let us give some heuristics before presenting the main results. In the majority dynamics on SBM, the agents are more stubborn to be influenced by others with a different opinion than in the Erdős-Rényi model. Suppose that we start from $|V_+^{(0)}| = n + \Delta$ and $|V_-^{(0)}| = n$, and we wish that the opinion $+$ wins eventually. Any agent with opinion $-$ will receive $\mathsf{Bin}(n + \Delta, q)$ many opinions $+$ and $\mathsf{Bin}(n - 1, p)$ many opinions $-$. After taking expectation, this yields the natural guess that when $(n + \Delta)q \geqslant (n - 1)p$ or (asymptotically) equivalently $\Delta > (p - q)n/q$, the opinion $+$ will win, which is not difficult to confirm. On the other hand, having $\Delta \geqslant 1$ already breaks the symmetry between different opinions. So we expect for both models that the opinion $+$ dominates at some threshold between $\Delta = 1$ and $\Delta = (p - q)n/q$, depending on the dependency of memories in the graph evolution.

Our first result shows that the Markovian model exhibits the power-of-one behavior. We denote by $L = L(p, q) > 0$ a large and explicitly computable constant depending only on $p, q$ that *may not be the same on each occurrence.*

**Theorem 1.** *Consider the Markovian model on* $\mathsf{SBM}(n + \Delta, n, p, q)$*, where* $0 \leqslant q < p \leqslant 1$*.*

(i) *Uniformly for* $\Delta > 0$ *and* $n \in \mathbb{N}$*, there exists* $L(p, q) > 0$ *such that*

$$\mathbb{P}[\mathscr{P}] \geqslant \frac{1}{2} + \frac{1}{L},$$

*i.e., the opinion* $+$ *wins eventually with probability at least* $\frac{1}{2} + \frac{1}{L}$*.*

(ii) *If* $\Delta(n) \to \infty$*, then* $\mathbb{P}[\mathscr{P}] \to 1$*, i.e., the opinion* $+$ *wins asymptotically almost surely.*

**Remark 4.** Let us note that for the Markovian model, the random walk evolves very slowly and the time till consensus will be exponentially increasing in $n$ for small $\Delta$ (e.g., for $\Delta$ with $(p - q)n/q - \Delta = \Omega(n)$, as can be seen from the proof of Theorem 1), and thus is not of great interest to us.

Compared to the Markovian model, the non-Markovian model exhibits a completely different behavior. Although possessing a similar form as the majority dynamics on Erdős-Rényi graphs studied in [TV20, BD22, SS22], the picture is still quite different.

(i) The block structure with $p > q$ makes the connection densities distinct between and within each block, and thus we are comparing binomial distributions with different means (see Remark 3), and the order of such differences needs to be controlled.

(ii) The dynamics halts for a large range of $\Delta$, due to the constraint $p > q$. For example, if at one round nobody changes opinion, the dynamics halts.

(iii) For $\Delta \geqslant 0$, with high probability there will not be any opinion change from $+$ to $-$ due to the significant difference of the binomial means (see Proposition 4.1 below), which is not the case when $p = q$.

Our main result can be stated as follows.

**Theorem 2.** *Consider the non-Markovian model on* $\mathsf{SBM}(n+\Delta, n, p, q)$ *where* $0 < q < p < 1$, *and the constant*

$$H = H(p, q) = \frac{\sqrt{p(2-p-q)}}{q}.$$

(i) *If* $\{d_n\}$ *is any sequence with* $d_n \to +\infty$ *and*

$$\Delta(n) \leqslant \left(\frac{p-q}{q}\right) n - H\left(\sqrt{\frac{3}{2}n\log n} + \sqrt{\frac{25n(\log\log n)^2}{24\log n}} + \sqrt{\frac{nd_n}{\log n}}\right), \qquad (2)$$

*then* $\mathbb{P}[\mathscr{T}] \to 1$, *i.e., the dynamics halts asymptotically almost surely.*

(ii) *For any sufficiently large constant* $L(p, q) > 0$, *if*

$$\Delta(n) \geqslant \left(\frac{p-q}{q}\right) n + L\sqrt{n\log n},$$

*then* $\mathbb{P}[\mathscr{P}_1] \to 1$, *i.e., the opinion* $+$ *wins on the first day asymptotically almost surely.*

(iii) *For any fixed constant* $\delta > 0$, *if*

$$\Delta(n) \geqslant \left(\frac{p-q}{q}\right) n - (H - \delta)\sqrt{n\log n},$$

*then* $\mathbb{P}[\mathscr{P}_2] \to 1$, *i.e., the opinion* $+$ *wins on the second day asymptotically almost surely.*

(iv) *For any sufficiently large constant* $L(p, q) > 0$, *if*

$$\Delta(n) \geqslant \left(\frac{p-q}{q}\right) n - H\left(\sqrt{n\log n} - \frac{3}{2}\sqrt{\frac{n(\log\log n)^2}{\log n}} - \sqrt{\frac{Ln}{\log n}}\right), \qquad (3)$$

*then* $\mathbb{P}[\mathscr{P}_3] \to 1$, *i.e., the opinion* $+$ *wins on the third day asymptotically almost surely.*

(v) *Asymptotically almost surely, the opinion* $-$ *will not win, i.e.,* $\mathbb{P}[\mathscr{M}] = o(1)$.

We remark that the rates at which the probabilities converge to 1 can be analyzed explicitly in the proofs. Observe that there is a transition between the phases $\mathbb{P}[\mathscr{T}] \to 1$ and $\mathbb{P}[\mathscr{P}] \to 1$. The leading term $(p-q)n/q$ for $\Delta$ where the phase transition takes place is given heuristically by Remark 3. We conjecture below that the subleading term is $-H\sqrt{n\log n}$.

**Conjecture 1.** *For any sufficiently large constant* $L(p, q) > 0$, *if*

$$\Delta \leqslant \left(\frac{p-q}{q}\right) n - H\sqrt{n\log n} - L\sqrt{\frac{n(\log\log n)^2}{\log n}},$$

*then* $\mathbb{P}[\mathscr{T}] \to 1$, *i.e., the majority dynamics halts asymptotically almost surely.*

6

In other words, at the critical point $\Delta_c = \left(\frac{p-q}{q}\right) n - H\sqrt{n \log n}$ within a window of size $O(\sqrt{n(\log \log n)^2 / \log n})$, there is a sharp phase transition for the outcome of the dynamics. On one side, the dynamics halts asymptotically almost surely. On the other side, we have a fast convergence to unanimity within three days. The difficulty here is to analyze the behavior of the dynamics for multiple days (since the first few days do not give sufficient information on whether the dynamics will halt), especially when the underlying graph evolves with time. Meanwhile, in the literature of non-Markovian models, the analysis of two days would typically suffice [BCO$^+$16, FKM20, Zeh20, SS22, BD22].

The precise form of conjecture 1 is motivated by the simpler case $p = 1$ below. We have focused on the case $p < 1$ in the non-Markovian model by now. Nevertheless, when $p = 1$, agents with the same opinion will always be connected, and the independence structure is more tractable. In this special case, we confirm Conjecture 1, establishing the sharp phase transition.

**Theorem 3.** *Consider the non-Markovian model on* $\mathsf{SBM}(n + \Delta, n, 1, q)$*, where* $0 < q < 1$*.*

*(i) For any sufficiently large constant $L(p, q) > 0$, if*

$$\Delta(n) \leqslant \left(\frac{1-q}{q}\right) n - \frac{\sqrt{1-q}}{q}\sqrt{n \log n} - L\sqrt{\frac{n(\log \log n)^2}{\log n}}, \qquad (4)$$

*then $\mathbb{P}[\mathscr{T}] \to 1$, i.e., the dynamics halts asymptotically almost surely.*

*(ii) For any fixed constant $\delta > 0$, if*

$$\Delta(n) \geqslant \left(\frac{1-q}{q}\right) n - \left(\frac{\sqrt{1-q}}{q} - \delta\right)\sqrt{n \log n},$$

*then $\mathbb{P}[\mathscr{P}_2] \to 1$, i.e., the opinion $+$ wins on the second day asymptotically almost surely.*

*(iii) For any sufficiently large constant $L(p, q) > 0$, if*

$$\Delta(n) \geqslant \left(\frac{1-q}{q}\right) n - \frac{\sqrt{1-q}}{q}\left(\sqrt{n \log n} - \sqrt{\frac{n(\log \log n)^2}{\log n}} - \sqrt{\frac{Ln}{\log n}}\right),$$

*then $\mathbb{P}[\mathscr{P}_3] \to 1$, i.e., the opinion $+$ wins on the third day asymptotically almost surely.*

**Remark 5.** Among the literature on majority dynamics, the assignment of initial opinions can be either deterministic or random. Models involving random initial data have been studied in [BCO$^+$16, FKM20], among many others. More precisely, the set $V_+^{(0)}$ (and hence $V_-^{(0)}$) is now determined by a sequence of i.i.d. random variables independent of everything else, where each vertex has probability $r \in (0, 1)$ holding opinion $+$ and $1 - r$ holding opinion $-$. In the framework of the Markovian model, if $r > 1/2$ then $\mathscr{P}$ holds asymptotically almost surely (in short, a.a.s.); if $r < 1/2$ then $\mathscr{M}$ holds a.a.s.; if $r = 1/2$ then both $\mathscr{P}$ and $\mathscr{M}$ hold with probability tending to $1/2$. For the non-Markovian model, as a consequence of Theorem 2 and the central limit theorem, if $r \geqslant p/(p+q)$ then $\mathscr{P}_2$ holds a.a.s.; if $r \leqslant q/(p+q)$ then $\mathscr{M}_2$ happens a.a.s.; otherwise, $\mathscr{T}$ holds a.a.s. This completely characterizes the behavior of our models under random initial conditions.

## 1.3 Notations and organization of the paper

For $n \in \mathbb{N}$, let $[n]$ denote the set of integers $\{1, \ldots, n\}$. We use small boldface letters to denote sequences. For a vector or a sequence $\mathbf{a} = (a_1, \ldots, a_n)$, we use $|\mathbf{a}| = \sum_{i=1}^{n} |a_i|$ to denote the $\ell_1$ norm. The $j$-th component of $\mathbf{a}$ is also denoted by $\mathbf{a}(j)$.

For $m, n \in \mathbb{N}$ and $p \in [0, 1]$, let $\mathsf{G}(n, p)$ denote the Erdős-Rényi graph and let $\mathsf{G}(m, n, p)$ denote a random bipartite graph with $m$ vertices on side and $n$ vertices on the other, each edge included independently with probability $p$.

For $\mu, \sigma \in \mathbb{R}$, let $\mathcal{N}(\mu, \sigma^2)$ be the normal distribution with mean $\mu$ and variance $\sigma^2$. For $n \in \mathbb{N}$ and $p \in [0, 1]$, we use $\mathsf{Bin}(n, p)$ to denote the binomial distribution with parameters $n$ and $p$. When a binomial distribution appears inside a probability operator $\mathbb{P}$, it should be interpreted as a binomial random variable with such a distribution independent of everything else, unless, when a certain distribution appears multiple times, they are interpreted as being equal instead of being independent.

The rest of the paper is organized as follows. In Sectio 2, we briefly describe the proof strategies which will be different for the two distinct models. In Section 3, we collect some auxiliary results, which include a summary of graph enumeration results of random graphs in Section 3.1 and nearly-optimal binomial tail bounds in Section 3.2. The complete proofs of the theorems are given in Section 4. Finally, we provide some numerical simulations in Section 5 and discuss some open questions in Section 6.

## 2 Proof Strategies

### 2.1 Markovian model

In the Markovian model, thanks to the resampling of the whole graph, at each step the evolution of the model can be treated as the first round with initialization given by the updated opinions from the previous step. This implies that the number of vertices holding opinion $+$, $\{|V_+^{(t)}|\}_{t \in \mathbb{N}_0}$, forms a Markovian random walk on $[0, 2n + \Delta] \cap \mathbb{Z}$. This reduces Theorem 1 to the analysis of a certain random walk on $\mathbb{Z}$. Such a random walk is symmetric around $n + \Delta/2$. Intuitively speaking, the walk is attracted by its two endpoints 0 and $2n + \Delta$. Thus to determine $\mathbb{P}[\mathscr{P}]$ we need to understand the behavior of the walk near the center $n + \Delta/2$. A crucial estimate, given by Lemma 4.1, states that near the center of the chain, the random walk rarely performs a move of length greater than one, and that the ratio of probabilities of moving right by one to that of moving left by one is controlled from below by some constant $1 + 1/L$, uniformly in $n$ and $\Delta$. Roughly speaking, this stems from the fact that the binomial tails are exponentially decreasing away from the mean. Theorem 1 then follows from Gambler's Ruin estimates, together with the observation that being close to an endpoint ensures the chain to reach the endpoint with high probability, which is given by Proposition 4.4.

### 2.2 Non-Markovian model

For the non-Markovian model, due to the dependency on the memory of previous steps, we need to track the evolution of graphs more carefully. The first step is to understand the joint distribution of the degrees of the vertices on the initial day. The marginal distribution of the degree of a fixed vertex is binomial, whereas the degrees are not independent for different neighbors. We bypass this difficulty by replacing the degrees with a simpler probabilistic model, in which the degrees of the vertices form a sequence of independent binomial random variables. This idea is based on graph

enumeration results of random graphs by McKay and Wormald [MW97], which roughly states that the distributions of the degrees in a random graph are approximately conditionally independent. We address the reduction to the independent degree model in Section 3.1.

We want to emphasize that although the previous work [SS22] also uses grpah enumeration techniques, our method is very different. In our work, graph enumerations are only used in the analysis for the initial day. Moreover, graph enumerations cannot be applied for the follow-up days in the process. This is because the dependency of past memories makes the edge connections highly correlated. The edge connections are hard to track along the process, and thus it is impossible to apply similar analysis for steps after day 1 as in the static Erdős-Rényi model.

We have mentioned that one feature of the non-Markovian model is that if at one step nobody changes opinion, then the model becomes stationary and nobody changes opinion anymore. Thus, a lower bound of $\mathbb{P}[\mathscr{T}]$ is given by the probability that the model halts at the first few steps. This establishes part (i) in Theorem 2, and similarly in the simpler case of Theorem 3 (i).

The route to proving (ii) to (v) of Theorem 2 is taken differently from the literature due to the intrinsic difference of our models. Since (ii) and (iii) are similar but simpler, we explain below the proof of part (iv). Part (v) will be a simple consequence of Proposition 4.1, which states that there are no vertices changing opinion from $+$ to $-$ a.a.s. Similar arguments also apply to Theorem 3 (ii) and (iii).

In short, the condition in Theorem 2 (iv) guarantees with high probability an increment on $|V_+^{(t)}|$ of $L\sqrt{n\log n}$ in the first day, $\delta n^{1/2+\delta}$ in the second, and making opinion $+$ winning in the third. The increment on the first day is purely marginal depending only on the SBM graph, which may be bounded explicitly using the conditioning structure of the number of neighbors after the reduction to the independent degree model. Given the increase on the first day, for a vertex $v \in V_-^{(1)}$ it receives new $+$ neighbors from the $L\sqrt{n\log n}$ ones, giving roughly $qL\sqrt{n\log n}$ connections. Thus, for those $v$ which has $-$ connections not exceeding $+$ connections by $qL\sqrt{n\log n}$, it will turn $+$, i.e., $v \in V_+^{(2)}$. This yields a good number of $\delta n^{1/2+\delta}$. On the third day we apply the same logic, and conclude via union bound that those $v \in V_-^{(2)}$ with the number of $-$ neighbors exceeding $+$ by $q\delta n^{1/2+\delta}$ are very few. Although this analysis generalizes to multiple days, it will only affect the lower order terms, and thus will not be considered in this paper.

# 3 Preliminaries

## 3.1 Reduction to the independent model

The swapping of opinions in the majority dynamics is based on the degrees of the vertices in the stochastic block model, but the distribution of the true degrees of an SBM is hard to analyze due to the constraint from the graph structure. To overcome this issue, as mentioned in the outline of proofs, we rely on the graph enumeration technique developed by Mckay and Wormald [MW90, MW97]. At a high level, their work implies that the degrees of random graphs look conditionally independent. These techniques apply to the non-Markovian model in the proofs of Theorems 2 and 3.

To make our paper self-contained, in this section we review some probabilistic models for the degree sequences, which will be used to give a quantitative version of the aforementioned high-level idea of graph enumerations. To begin with, let us define the domains where the degree sequences are defined, following [MW97].

**Definition 5** (Degree sequence domains). Denote $I_n = \{0, \ldots, n-1\}^n$. Let $E_n$ be the even sum sequences in $I_n$. The elements of these sets are typically denoted by the small bold letter $\mathbf{d}$. For

the bipartite setting, similarly denote $I_{m,n} = \{0, \ldots, n\}^m \times \{0, \ldots, m\}^n$. Let $E_{m,n}$ be the sequences with equal sums on both sides. The elements of these sets are typically denoted by small boldface letters $\mathbf{s}$ of length $m$ and $\mathbf{t}$ of length $n$. The corresponding random variables will be denoted by capital boldface letters.

The distribution of the true degree sequence in $\mathsf{G}(n, p)$ and $\mathsf{G}(m, n, p)$ is denoted as follows.

**Definition 6** (True degree models). Let $\mathcal{D}_p^n$ be the degree sequence distribution of the Erdős-Rényi graph $\mathsf{G}(n, p)$, which is a random variable supported on $E_n \subset I_n$. For the bipartite setting, let $\mathcal{D}_p^{m,n}$ be the degree sequence distribution of the random bipartite graph $\mathsf{G}(m, n, p)$, which is a random variable supported on $E_{m,n} \subset I_{m,n}$.

For approximations of the true degree model, we introduce the following models.

**Definition 7** (Independent degree models). Let $\mathcal{B}_p^n$ be the distribution of $n$ independent copies of $\mathsf{Bin}(n - 1, p)$ random variables, which is supported on $I_n$. For the bipartite setting, let $\mathcal{B}_p^{m,n}$ be the distribution of $m$ independent $\mathsf{Bin}(n, p)$ variables and $n$ independent $\mathsf{Bin}(m, p)$ variables, which is supported on $I_{m,n}$.

**Definition 8** (Conditioned degree models). Let $\mathcal{E}_p^n$ be the distribution of $\mathcal{B}_p^n$ conditioned on having even sum, which is supported on $E_n$.

**Definition 9** (Integrated degree models). Let $\mathcal{I}_p^n$ be the distribution defined as follows: First sample $p' \sim \mathcal{N}(p, \frac{p(1-p)}{n(n-1)})$, conditional on $p' \in (0, 1)$; then sample from $\mathcal{E}_{p'}^n$.

Using the models introduced above, we now state the following necessary preliminary result.

**Theorem 4** ([MW90, Theorem 3] and [MW97, Theorem 3.6]). *There exists $c > 0$ so that the following is true. Let $n \geqslant 2$ and suppose that $(\log n)^{-1/4} \leqslant p \leqslant 1 - (\log n)^{-1/4}$. There is an event $B_p^n \subset I_n$ such that $\mathbb{P}_{\mathcal{D}_p^n}[B_p^n] = n^{-\omega(1)}$ and uniformly for all $\mathbf{d} \in I_n \backslash B_p^n$ we have*

$$\mathbb{P}_{\mathcal{D}_p^n}[\mathbf{D} = \mathbf{d}] = \left(1 + O(n^{-c})\right) \mathbb{P}_{\mathcal{I}_p^n}[\mathbf{D} = \mathbf{d}].$$

In the remainder of this paper, when dealing with probabilities concerning random graphs, the notation $\mathbb{P}$ (without mentioning the probability model explicitly) represents the probability measure of the true degree models. For example, the conclusion of Theorem 4 can be written as

$$\mathbb{P}[\mathbf{D} = \mathbf{d}] = \left(1 + O(n^{-c})\right) \mathbb{P}_{\mathcal{I}_p^n}[\mathbf{D} = \mathbf{d}].$$

Note that the degree sequence in $\mathsf{SBM}(m, n, p, q)$ is sampled from three independent subgraphs: $\mathsf{G}(m, p)$, $\mathsf{G}(n, p)$ and $\mathsf{G}(m, n, q)$. Using Theorem 4, when considering the marginal probabilities on a single block, we can replace the true degrees of the Erdős-Rényi subgraph with the integrated degree model, and henceforth reduce it to the independent degree model. Specifically, we have the following result.

**Lemma 3.1.** *For any sufficiently large fixed $L > 0$, we have*

$$\mathbb{P}_{\mathcal{D}_p^{n+\Delta}, \mathcal{B}_q^{n+\Delta,n}}\left[|V_+^{(0)} \cap V_+^{(1)}| = x\right]$$
$$= \left(1 + O(n^{-c})\right) \int_R \mathbb{P}_{\mathcal{B}_{r_1}^{n+\Delta}, \mathcal{B}_q^{n+\Delta,n}}\left[|V_+^{(0)} \cap V_+^{(1)}| = x\right] d\mu_1(r_1) + O(n^{-\omega(1)}),$$

*and*

$$\mathbb{P}_{\mathcal{D}_p^n, \mathcal{B}_q^{n+\Delta,n}} \left[ |V_-^{(0)} \cap V_-^{(1)}| = y \right]$$

$$= \left(1 + O(n^{-c})\right) \int_R \mathbb{P}_{\mathcal{B}_{r_2}^n, \mathcal{B}_q^{n+\Delta,n}} \left[ |V_-^{(0)} \cap V_-^{(1)}| = y \right] \mathrm{d}\mu_2(r_2) + O(n^{-\omega(1)}).$$

*where the measure $\mu_1$ is the distribution $\mathcal{N}\left(p, \frac{p(1-p)}{(n+\Delta)(n+\Delta-1)}\right)$, $\mu_2$ is the distribution $\mathcal{N}\left(p, \frac{p(1-p)}{n(n-1)}\right)$, and the integral domain is given by*

$$R := \left\{ r \in [0,1] : |r - p| \leqslant \frac{L \log n}{n} \right\}. \tag{5}$$

*Proof.* For a given graph, the swapping of opinions purely depends on the degree information. This implies that the sizes of the swapped vertices, as random variables, are measurable with respect to the degree sequences. Note that the degree sequences are sampled from three independent random graphs: two Erdős-Rényi graphs $\mathsf{G}(n+\Delta, p)$, $\mathsf{G}(n,p)$, and a random bipartite graph $\mathsf{G}(n+\Delta, n, q)$, i.e., the randomness of this event comes from the true degree models $\mathcal{D}_p^{n+\Delta}$, $\mathcal{D}_p^n$ and $\mathcal{D}_q^{n+\Delta,n}$ independently. Since we are interested in the marginal behavior of a single block, by Theorem 4, we can replace the true degree models $\mathcal{D}_p^{n+\Delta}$ and $\mathcal{D}_p^n$ with the integrated degree models $\mathcal{I}_p^{n+\Delta}$ and $\mathcal{I}_p^n$ up to a $1 + O(n^{-c})$ multiplicative factor and an additive error term of size $O(n^{-\omega(1)})$.

For the two Erdős-Rényi subgraphs, let $\mathbf{d}_1 \in I_{n+\Delta}$ be the degree sequence of length $n + \Delta$ and $\mathbf{d}_2 \in I_n$ be the degree sequence of length $n$. For the bipartite subgraph, we use $(\mathbf{s}, \mathbf{t}) \in I_{n+\Delta,n}$ to denote the degree sequences of length $n + \Delta$ and $n$, respectively. We treat the swapped sets $V_+^{(1)}$ and $V_-^{(1)}$ as functions of $(\mathbf{d}_1, \mathbf{d}_2, (\mathbf{s}, \mathbf{t}))$, and extend these functions in the obvious way if the total sum in $I_{n+\Delta}$ or $I_n$ is not even, or the sums on both sides of $I_{n+\Delta,n}$ do not match.

Using these notations, by Theorem 4, we have

$$\mathbb{P}_{\mathcal{D}_p^{n+\Delta}, \mathcal{B}_q^{n+\Delta,n}} \left[ |V_+^{(0)} \cap V_+^{(1)}| = x \right]$$

$$= \left(1 + O(n^{-c})\right) \mathbb{P}_{\mathcal{I}_p^{n+\Delta}, \mathcal{B}_q^{n+\Delta,n}} \left[ |V_+^{(0)} \cap V_+^{(1)}| = x \right] + O(n^{-\omega(1)}). \tag{6}$$

*and*

$$\mathbb{P}_{\mathcal{D}_p^n, \mathcal{B}_q^{n+\Delta,n}} \left[ |V_-^{(0)} \cap V_-^{(1)}| = y \right] = \left(1 + O(n^{-c})\right) \mathbb{P}_{\mathcal{I}_p^n, \mathcal{B}_q^{n+\Delta,n}} \left[ |V_-^{(0)} \cap V_-^{(1)}| = y \right] + O(n^{-\omega(1)}). \tag{7}$$

By the definition of integrated degree models, we further have

$$\mathbb{P}_{\mathcal{I}_p^{n+\Delta}, \mathcal{B}_q^{n+\Delta,n}} \left[ |V_+^{(0)} \cap V_+^{(1)}| = x \right] = \frac{1}{\int_{[0,1]} \mathrm{d}\mu_1(r_1)} \int_{[0,1]} \mathbb{P}_{\mathcal{E}_{r_1}^{n+\Delta}, \mathcal{B}_q^{n+\Delta,n}} \left[ |V_+^{(0)} \cap V_+^{(1)}| = x \right] \mathrm{d}\mu_1(r_1),$$

*and*

$$\mathbb{P}_{\mathcal{I}_p^n, \mathcal{B}_q^{n+\Delta,n}} \left[ |V_-^{(0)} \cap V_-^{(1)}| = y \right] = \frac{1}{\int_{[0,1]} \mathrm{d}\mu_2(r_2)} \int_{[0,1]} \mathbb{P}_{\mathcal{E}_{r_2}^n, \mathcal{B}_q^{n+\Delta,n}} \left[ |V_-^{(0)} \cap V_-^{(1)}| = y \right] \mathrm{d}\mu_2(r_2).$$

Consider the random variables

$$Z_1 \sim \mathcal{N}\left(p, \frac{p(1-p)}{(n+\Delta)(n+\Delta-1)}\right), \quad Z_2 \sim \mathcal{N}\left(p, \frac{p(1-p)}{n(n-1)}\right).$$

11

The Gaussian tail bound gives that for $i = 1, 2$,

$$\mathbb{P}\left[|Z_i - p| \leqslant \frac{L \log n}{n}\right] = 1 - O(n^{-\omega(1)}).$$

This yields

$$\mathbb{P}_{\mathcal{I}_p^{n+\Delta}, \mathcal{B}_q^{n+\Delta, n}}\left[|V_+^{(0)} \cap V_+^{(1)}| = x\right] = \int_R \mathbb{P}_{\mathcal{E}_{r_1}^{n+\Delta}, \mathcal{B}_q^{n+\Delta, n}}\left[|V_+^{(0)} \cap V_+^{(1)}| = x\right] d\mu_1(r_1) + O(n^{-\omega(1)}),$$

and

$$\mathbb{P}_{\mathcal{I}_p^n, \mathcal{B}_q^{n+\Delta, n}}\left[|V_-^{(0)} \cap V_-^{(1)}| = y\right] = \int_R \mathbb{P}_{\mathcal{E}_{r_2}^n, \mathcal{B}_q^{n+\Delta, n}}\left[|V_-^{(0)} \cap V_-^{(1)}| = y\right] d\mu_2(r_2) + O(n^{-\omega(1)}),$$

where $R$ is given by (5).

Finally, we remove the evenness constraints in the conditioned models $\mathcal{E}_{r_1}^{n+\Delta}$ and $\mathcal{E}_{r_2}^n$. Note that the Bayes' rule implies

$$\mathbb{P}_{\mathcal{E}_{r_1}^{n+\Delta}, \mathcal{B}_q^{n+\Delta, n}}\left[|V_+^{(0)} \cap V_+^{(1)}| = x\right] = \frac{\mathbb{P}_{\mathcal{B}_{r_1}^{n+\Delta}, \mathcal{B}_q^{n+\Delta, n}}\left[|V_+^{(0)} \cap V_+^{(1)}| = x, |\mathbf{D}_+| \in 2\mathbb{Z}\right]}{\mathbb{P}_{\mathcal{B}_{r_1}^{n+\Delta}}\left[|\mathbf{D}_+| \in 2\mathbb{Z}\right]},$$

and

$$\mathbb{P}_{\mathcal{E}_{r_2}^n, \mathcal{B}_q^{n+\Delta, n}}\left[|V_-^{(0)} \cap V_-^{(1)}| = y\right] = \frac{\mathbb{P}_{\mathcal{B}_{r_2}^n, \mathcal{B}_q^{n+\Delta, n}}\left[|V_-^{(0)} \cap V_-^{(1)}| = y, |\mathbf{D}_-| \in 2\mathbb{Z}\right]}{\mathbb{P}_{\mathcal{B}_{r_2}^n}\left[|\mathbf{D}_-| \in 2\mathbb{Z}\right]},$$

where $\mathbf{D}_+$ and $\mathbf{D}_-$ are the degree sequences of $V_+^{(0)}$ and $V_-^{(0)}$, respectively. Using the arguments in [SS22, Equation (2.4)], we have

$$\frac{\mathbb{P}_{\mathcal{B}_{r_1}^{n+\Delta}, \mathcal{B}_q^{n+\Delta, n}}\left[|V_+^{(0)} \cap V_+^{(1)}| = x, |\mathbf{D}_+| \in 2\mathbb{Z}\right]}{\mathbb{P}_{\mathcal{B}_{r_1}^{n+\Delta}}\left[|\mathbf{D}_+| \in 2\mathbb{Z}\right]}$$
$$= \frac{(\frac{1}{2} + O(\exp(-n)))\mathbb{P}_{\mathcal{B}_{r_1}^{n+\Delta}, \mathcal{B}_q^{n+\Delta, n}}\left[|V_+^{(0)} \cap V_+^{(1)}| = x\right] + O\left(\exp(-\Omega(n))\right)}{\frac{1}{2} + O(\exp(-n))}$$
$$= \mathbb{P}_{\mathcal{B}_{r_1}^{n+\Delta}, \mathcal{B}_q^{n+\Delta, n}}\left[|V_+^{(0)} \cap V_+^{(1)}| = x\right] + O\left(\exp(-\Omega(n))\right),$$

and similarly

$$\frac{\mathbb{P}_{\mathcal{B}_{r_2}^n, \mathcal{B}_q^{n+\Delta, n}}\left[|V_-^{(0)} \cap V_-^{(1)}| = y, |\mathbf{D}_-| \in 2\mathbb{Z}\right]}{\mathbb{P}_{\mathcal{B}_{r_2}^n}\left[|\mathbf{D}_-| \in 2\mathbb{Z}\right]} = \mathbb{P}_{\mathcal{B}_{r_2}^n, \mathcal{B}_q^{n+\Delta, n}}\left[|V_-^{(0)} \cap V_-^{(1)}| = y\right] + O\left(\exp(-\Omega(n))\right).$$

Plugging back into (6) and (7) completes the proof. $\square$

## 3.2 Tail estimates of binomial distributions

For the proofs of the main theorems, we will frequently use the marginal probabilities that the opinion of a certain vertex flips at the first step in majority dynamics on $\mathsf{SBM}(n + \Delta, n, p, q)$, defined by

$$p_{-+} := \mathbb{P}\left[\mathsf{Bin}(n + \Delta, q) > \mathsf{Bin}(n - 1, p)\right], \quad p_{+-} := \mathbb{P}\left[\mathsf{Bin}(n, q) > \mathsf{Bin}(n + \Delta - 1, p)\right]. \quad (8)$$

In this section, we collect some technical results concerning the tail behavior of binomial random variables and give upper and lower bounds for quantities of the form (8). Throughout we consider two fixed constants $p, q \in (0,1)$ with $p > q$, and $\widetilde{p} \in [p - L(\log n)/n, p + L(\log n)/n]$. Define $\widetilde{p}_{-+} := \mathbb{P}[\mathsf{Bin}(n+\Delta, q) > \mathsf{Bin}(n-1, \widetilde{p})]$ similarly as in (8). Recall Hoeffding's inequality that for $k \leqslant n\widetilde{p}$,

$$\mathbb{P}[\mathsf{Bin}(n, \widetilde{p}) \leqslant k] \leqslant L \exp\left(-2n\left(p - \frac{k}{n}\right)^2\right). \tag{9}$$

The following Lemma gives upper and lower bounds for the probability in (9) in the case $np - k = O(\sqrt{n \log n})$. For simplicity of notations, let us define

$$\Delta'(n) := \left(\frac{p-q}{q}\right)n - \Delta(n)$$

**Lemma 3.2.** *Assume that $\Delta'(n) = O(\sqrt{n \log n})$, then there exists $M = M(\Delta', p, q) > 0$ so that*

$$\frac{1}{M\sqrt{\log n}} \exp\left(-\frac{C\Delta'^2(n+\Delta)}{n^2}\right) \leqslant \mathbb{P}[\mathsf{Bin}(n+\Delta, q) > n\widetilde{p}] \leqslant M \exp\left(-\frac{C\Delta'^2(n+\Delta)}{n^2}\right), \tag{10}$$

*where $C = C(p, q) = q^3/(2(1-q)p^2)$.*

*Proof.* Recall from [Ash12, Lemma 4.7.2] the classical tail bounds for binomial distribution (here $p$ may depend on $n$):

$$\frac{1}{L\sqrt{n}} \exp\left(-(n+\Delta)D\left(\left.\frac{np}{n+\Delta}\right\| q\right)\right) \leqslant \mathbb{P}[\mathsf{Bin}(n+\Delta, q) = np] \leqslant \mathbb{P}[\mathsf{Bin}(n+\Delta, q) \geqslant np]$$

$$\leqslant L \exp\left(-(n+\Delta)D\left(\left.\frac{np}{n+\Delta}\right\| q\right)\right), \tag{11}$$

where the Kullback-Leibler divergence $D(\cdot \| \cdot)$ is given by

$$D(a \| p) := a\log\left(\frac{a}{p}\right) + (1-a)\log\left(\frac{1-a}{1-p}\right).$$

Here and later, for simplicity we may assume $np$ as well as any other quantities that are $\omega(1)$ to be integers. Applying the floor or ceiling functions will not change the final results.

Using the inequalities $x - x^2/2 \leqslant \log(1+x) \leqslant x - x^2/2 + Lx^3$ and $-x - x^2/2 - Lx^3 \leqslant \log(1-x) \leqslant -x - x^2/2$ for $0 < x < 1$, we compute

$$D\left(\left.\frac{n\widetilde{p}}{n+\Delta}\right\| q\right)$$

$$= \frac{np}{\frac{p}{q}n - \Delta'} \log\left(\frac{\frac{np}{q}}{\frac{p}{q}n - \Delta'}\right) + \left(1 - \frac{np}{\frac{p}{q}n - \Delta'}\right) \log\left(\frac{1 - \frac{np}{\frac{p}{q}n - \Delta'}}{1 - q}\right) + o\left(\frac{1}{n}\right)$$

$$\geqslant \frac{np}{\frac{p}{q}n - \Delta'}\left(\frac{\Delta'}{\frac{p}{q}n - \Delta'} - \frac{\Delta'^2}{2(\frac{p}{q}n - \Delta')^2}\right)$$

$$+ \left(1 - \frac{np}{\frac{p}{q}n - \Delta'}\right)\left(-\frac{q\Delta'}{(1-q)(\frac{p}{q}n - \Delta')} - \frac{(q\Delta')^2}{2(1-q)^2(\frac{p}{q}n - \Delta')^2} - \frac{L(q\Delta')^3}{(1-q)^3(\frac{p}{q}n - \Delta')^3}\right) + o\left(\frac{1}{n}\right)$$

$$= \left(\frac{q^3}{2(1-q)p^2}\right)\frac{\Delta'^2}{n^2} + o\left(\frac{1}{n}\right),$$

13

where we used in the last step that $\Delta'(n) = o(n^{2/3})$. Similarly,

$$
\begin{aligned}
D\left(\left.\frac{n\widetilde{p}}{n+\Delta}\right\| q\right) \leqslant &\ \frac{np}{\frac{p}{q}n - \Delta'}\left(\frac{\Delta'}{\frac{p}{q}n - \Delta'} - \frac{\Delta'^2}{2(\frac{p}{q}n - \Delta')^2} + \frac{L\Delta'^3}{(\frac{p}{q}n - \Delta')^3}\right) \\
&+ \left(1 - \frac{np}{\frac{p}{q}n - \Delta'}\right)\left(-\frac{q\Delta'}{(1-q)(\frac{p}{q}n - \Delta')} - \frac{(q\Delta')^2}{2(1-q)^2(\frac{p}{q}n - \Delta')^2}\right) + o\left(\frac{1}{n}\right) \\
= &\ \left(\frac{q^3}{2(1-q)p^2}\right)\frac{\Delta'^2}{n^2} + o\left(\frac{1}{n}\right).
\end{aligned}
$$

Plugging these estimates into (11) gives the desired upper bound

$$
\mathbb{P}[\mathsf{Bin}(n+\Delta, q) \geqslant n\widetilde{p}] \leqslant L\exp\left(-\frac{C\Delta'^2(n+\Delta)}{n^2}\right)
$$

and

$$
\mathbb{P}[\mathsf{Bin}(n+\Delta, q) = n\widetilde{p}] \geqslant \frac{1}{L\sqrt{n}}\exp\left(-\frac{C\Delta'^2(n+\Delta)}{n^2}\right). \tag{12}
$$

Let us refine the lower bound (12) to get a lower bound for $\mathbb{P}[\mathsf{Bin}(n+\Delta, q) \geqslant n\widetilde{p}]$. Note that

$$
\begin{aligned}
\frac{\mathbb{P}\left[\mathsf{Bin}(n+\Delta, q) = n\widetilde{p} + \sqrt{\frac{n}{\log n}}\right]}{\mathbb{P}[\mathsf{Bin}(n+\Delta, q) = n\widetilde{p}]} &= \left(\frac{q}{1-q}\right)^{\sqrt{\frac{n}{\log n}}}\frac{\binom{n+\Delta}{n\widetilde{p}+\sqrt{\frac{n}{\log n}}}}{\binom{n+\Delta}{n\widetilde{p}}} \\
&\geqslant \left(\frac{q(n+\Delta - n\widetilde{p} - \sqrt{\frac{n}{\log n}})}{(1-q)n\widetilde{p}}\right)^{\sqrt{\frac{n}{\log n}}} \\
&\geqslant \left(1 - M\sqrt{\frac{\log n}{n}}\right)^{\sqrt{\frac{n}{\log n}}} \geqslant \frac{1}{M}, \tag{13}
\end{aligned}
$$

where the constant $M$ may not be the same on each occurrence. By unimodality of the probability mass function of the binomial distribution and (12),

$$
\begin{aligned}
\mathbb{P}[\mathsf{Bin}(n+\Delta, q) \geqslant n\widetilde{p}] &\geqslant \mathbb{P}\left[\mathsf{Bin}(n+\Delta, q) = n\widetilde{p} + \sqrt{\frac{n}{\log n}}\right]\sqrt{\frac{n}{\log n}} \\
&\geqslant \frac{1}{M\sqrt{n}}\exp\left(-\frac{C\Delta'^2(n+\Delta)}{n^2}\right)\sqrt{\frac{n}{\log n}} \\
&\geqslant \frac{1}{M\sqrt{\log n}}\exp\left(-\frac{C\Delta'^2(n+\Delta)}{n^2}\right).
\end{aligned}
$$

This completes the proof. $\qquad\square$

We now apply Lemma 3.2 to obtain estimates on the (fundamentally important) probability that one binomial random variable is larger than the other.

**Lemma 3.3.** *Assume that $\Delta' = O(\sqrt{n\log n})$, then there exists $M = M(\Delta', p, q) > 0$ so that*

$$
\frac{1}{M(\log n)}\exp\left(-\frac{C'\Delta'^2}{n}\right) \leqslant \widetilde{p}_{-+} \leqslant M(\log n)\exp\left(-\frac{C'\Delta'^2}{n}\right),
$$

*where $C' = C'(p, q)$ is given by*

$$C'(p, q) = \frac{q^2}{2p(2 - p - q)}. \tag{14}$$

*Proof.* Denote by $K = K(n, \widetilde{p}, q, \Delta) \in \mathbb{Z} \cap [(n + \Delta)q, n\widetilde{p}]$ that solves the minimization problem

$$\min_{K \in \mathbb{Z} \cap [(n+\Delta)q, n\widetilde{p}]} \left( \frac{1}{2(1 - q)} \frac{q(n + \Delta)(K - (n + \Delta)q)^2}{K^2} + \frac{1}{2\widetilde{p}} \frac{(1 - \widetilde{p})n(n - K - n(1 - \widetilde{p}))^2}{(n - K)^2} \right)$$

and $C(n, \widetilde{p}, q, \Delta)$ the attained minimum. One checks using $\Delta'(n) = O(\sqrt{n \log n})$ that

$$C(n, \widetilde{p}, q, \Delta) = \min_{K \in [(n+\Delta)q, np]} \left( \frac{np(K - (n + \Delta)q)^2}{2(1 - q)(np)^2} + \frac{(1 - p)n(n - K - n(1 - p))^2}{2p(n(1 - p))^2} \right) + O(1)$$
$$= C'(p, q) \frac{\Delta'^2}{n} + O(1)$$

where $C'(p, q)$ is given by (14), and this holds uniformly for $\widetilde{p} \in [p - L(\log n)/n, p + L(\log n)/n]$. We have by using independence and (10) that

$$\widetilde{p}_{-+} \geqslant \mathbb{P}[\mathsf{Bin}(n + \Delta, q) \geqslant K] \, \mathbb{P}[\mathsf{Bin}(n - 1, \widetilde{p}) \leqslant K]$$
$$\geqslant \frac{1}{M \log n} \exp \left( -\frac{1}{2(1 - q)} \frac{q(n + \Delta)(K - (n + \Delta)q)^2}{K^2} - \frac{1}{2\widetilde{p}} \frac{(1 - \widetilde{p})n(n - K - n(1 - \widetilde{p}))^2}{(n - K)^2} \right)$$
$$\geqslant \frac{1}{M \log n} \exp \left( -\frac{C'\Delta'^2}{n} \right).$$

This gives the lower bound as desired.

For the upper bound, since $\Delta' = O(\sqrt{n \log n})$, we may chop the interval $[(n + \Delta)q, (n - 1)\widetilde{p}]$ into at most $M \log n$ pieces of lengths $\sqrt{n/\log n}$. This gives

$$\widetilde{p}_{-+} \leqslant \sum_{j=1}^{M \log n} \mathbb{P} \left[ \mathsf{Bin}(n - 1, \widetilde{p}) < j\sqrt{\frac{n}{\log n}} + (n + \Delta)q \right] \mathbb{P} \left[ \mathsf{Bin}(n + \Delta, q) > (j - 1)\sqrt{\frac{n}{\log n}} + (n + \Delta)q \right]$$
$$+ \mathbb{P}[\mathsf{Bin}(n - 1, \widetilde{p}) < (n + \Delta)q] + \mathbb{P}[\mathsf{Bin}(n + \Delta, q) > (n - 1)\widetilde{p}].$$

By the same arguments as in (13), the first term is bounded by

$$\sum_{j=1}^{M \log n} \mathbb{P} \left[ \mathsf{Bin}(n - 1, \widetilde{p}) < j\sqrt{\frac{n}{\log n}} + (n + \Delta)q \right] \mathbb{P} \left[ \mathsf{Bin}(n + \Delta, q) > (j - 1)\sqrt{\frac{n}{\log n}} + (n + \Delta)q \right]$$
$$\leqslant M \sum_{j=1}^{M \log n} \mathbb{P} \left[ \mathsf{Bin}(n - 1, \widetilde{p}) < j\sqrt{\frac{n}{\log n}} + (n + \Delta)q \right] \mathbb{P} \left[ \mathsf{Bin}(n + \Delta, q) > j\sqrt{\frac{n}{\log n}} + (n + \Delta)q \right]$$
$$\leqslant M \sum_{j=1}^{M \log n} \exp(-C(n, \widetilde{p}, q, \Delta))$$
$$\leqslant M(\log n) \exp \left( -\frac{C'\Delta'^2}{n} \right),$$

where the constant $M$ may not be the same on each occurrence. On the other hand, it is easy to check using Lemma 3.2 that the rest two terms satisfy

$$\mathbb{P}[\mathsf{Bin}(n-1,\widetilde{p}) < (n+\Delta)q] + \mathbb{P}[\mathsf{Bin}(n+\Delta,q) > (n-1)\widetilde{p}] \leqslant M(\log n)\exp\left(-\frac{C'\Delta'^2}{n}\right).$$

This finishes the proof of the upper bound. $\qquad\square$

The following lemmas deal with probabilities that a binomial random variable lies in a (random) interval, which will be useful for giving sufficient conditions for the dynamics to halt.

**Lemma 3.4.** *Consider a sequence $a_n = o(\sqrt{n/\log n})$. Suppose that $\Delta' \geqslant \delta\sqrt{n\log n}$ for some $\delta > 0$. Then there exists $M = M(\delta, p, q) > 0$ such that*

$$\frac{a_n(\log n)^{1/2}}{M\sqrt{n}}\exp\left(-\frac{C'\Delta'^2}{n}\right) \leqslant \mathbb{P}[\mathsf{Bin}(n+\Delta,q) \leqslant \mathsf{Bin}(n-1,\widetilde{p}) \leqslant \mathsf{Bin}(n+\Delta,q) + a_n]$$

$$\leqslant \frac{Ma_n(\log n)^{3/2}}{\sqrt{n}}\exp\left(-\frac{C'\Delta'^2}{n}\right).$$

*Proof.* Using similar arguments as in Lemma 3.3 in the second line, we have

$$\mathbb{P}[\mathsf{Bin}(n+\Delta,q) \leqslant \mathsf{Bin}(n-1,\widetilde{p}) \leqslant \mathsf{Bin}(n+\Delta,q) + a_n]$$

$$= (1+o(1)) \sum_{j=(n+\Delta)q+\sqrt{n\log n}/M}^{(n-1)\widetilde{p}-\sqrt{n\log n}/M} \mathbb{P}[\mathsf{Bin}(n-1,\widetilde{p}) = j]\mathbb{P}[j - a_n \leqslant \mathsf{Bin}(n+\Delta,q) \leqslant j]$$

$$\leqslant Ma_n\sqrt{\frac{\log n}{n}} \sum_{j=(n+\Delta)q+\sqrt{n\log n}/M}^{(n-1)\widetilde{p}-\sqrt{n\log n}/M} \mathbb{P}[\mathsf{Bin}(n-1,\widetilde{p}) = j]\mathbb{P}[\mathsf{Bin}(n+\Delta,q) \geqslant j]$$

$$\leqslant Ma_n\sqrt{\frac{\log n}{n}}\ \widetilde{p}_{-+}$$

$$\leqslant \frac{Ma_n(\log n)^{3/2}}{\sqrt{n}}\exp\left(-\frac{C'\Delta'^2}{n}\right)$$

where the last step is a consequence of Lemma 3.3. The lower bound is similar. $\qquad\square$

**Lemma 3.5.** *Consider sequences $a_n = o(\sqrt{n/\log n})$, $b_n = o(\sqrt{n\log n})$, $c_n = o(\sqrt{n\log n})$. Suppose that $\Delta' \geqslant \delta\sqrt{n\log n}$ for some constant $\delta > 0$. Then there exists $M$ sufficiently large (depending on $\delta$ the choice of the above sequences but not on $n$) such that for $n$ large enough,*

$$\mathbb{P}[n+b_n - a_n \leqslant \mathsf{Bin}(n+\Delta+c_n,q) \leqslant n+b_n] \leqslant \frac{a_n(\log n)^M}{\sqrt{n}}\exp\left(-\frac{C'\Delta'^2}{n}\right).$$

*Proof.* This follows from similar arguments as in the proof of Lemma 3.4, using Lemma 3.2 instead of Lemma 3.3. The multiplicative constant $M$ in Lemma 3.4 disappears by choosing here $M$ large enough. $\qquad\square$

# 4 Proofs of the main results

## 4.1 Sufficient conditions for consensus

For the non-Markovian model, as mentioned previously, the condition $p > q$ ensures that there would be no change of opinion from $+$ to $-$ with high probability if we start with an advantage of the opinion $+$. To illustrate this fact, for $t \in \mathbb{N}$, we define $\mathcal{N}_t := \{V_-^{(t)} \cap V_+^{(t-1)} \neq \emptyset\}$, the event that there exists an opinion change from $+$ to $-$ at time $t$. Let also $\mathcal{N} := \cup_{t \in \mathbb{N}} \mathcal{N}_t$.

**Proposition 4.1.** *Consider the non-Markovian model on* $\mathsf{SBM}(n + \Delta, n, p, q)$ *where* $0 < q < p \leqslant 1$ *and* $\Delta \geqslant 0$. *It holds that* $\mathbb{P}[\mathcal{N}] \to 0$.

*Proof.* It is straightforward to show using union bound and (9) that

$$\mathbb{P}[\mathcal{N}_1] \leqslant n p_{+-} \leqslant n \mathbb{P}\left[\mathsf{Bin}(n, q) > \mathsf{Bin}(n-1, p)\right] \leqslant L \exp(-n/L).$$

More generally, on the event $(\cup_{s=1}^{t-1} \mathcal{N}_s)^c$ we have that $|V_-^{(t-1)}| \leqslant n$, so that

$$\mathbb{P}\left[\mathcal{N}_t \middle| \left(\bigcup_{s=1}^{t-1} \mathcal{N}_s\right)^c\right] \leqslant n \mathbb{P}\left[\mathsf{Bin}(n, q) > \mathsf{Bin}(n, p)\right] \leqslant L \exp(-n/L).$$

Note also that

$$\mathbb{P}\left[\bigcup_{t > n} \mathcal{N}_t \middle| \left(\bigcup_{s=1}^{n} \mathcal{N}_s\right)^c\right] = 0,$$

because if there is no opinion change from $+$ to $-$ in the first $n$ steps, then it must be that the dynamics halts or the opinion $+$ wins. Combining the above yields that $\mathbb{P}[\mathcal{N}] \leqslant L \exp(-n/L)$. $\square$

We first analyze some simple sufficient conditions for the dynamics to win in the next day.

**Proposition 4.2.** *Let* $\delta > 0$ *and* $t \in \mathbb{N}_0$. *Recall that* $\Delta' = (p - q)n/q - \Delta$. *For the non-Markovian model with* $\Delta' = o(n^{1/2+\delta})$, *it holds that*

$$\mathbb{P}[\mathscr{P}_{t+1}] \geqslant \mathbb{P}\left[|V_-^{(t-1)} \cap V_+^{(t)}| \geqslant \delta n^{1/2+\delta}\right] - o(1).$$

*Proof.* For a vertex $v$, denote by $v_+^{(t)}, v_-^{(t)}$ the number of neighbors of $v$ in $V_+^{(t)}, V_-^{(t)}$ respectively for $t \in \mathbb{N}$. First, we intersect with the event $\mathcal{N}^c$ that there is no opinion changing from $+$ to $-$ throughout the dynamics, thus removing a set of probability $o(1)$ by Proposition 4.1. Next, we intersect with the event

$$\mathscr{E} := \left\{ v_-^{(0)} \leqslant np + n^{(\delta+1)/2} \text{ and } v_+^{(0)} \geqslant (n + \Delta)q - n^{(\delta+1)/2} \text{ for any } v \in V_-^{(0)} \right\}.$$

By (9) and the union bound, it holds that $\mathbb{P}[\mathscr{E}^c] = o(1)$. Therefore, it suffices to prove

$$\mathbb{P}\left[\mathscr{P}_{t+1}^c \cap \mathscr{E} \cap \mathcal{N}^c \cap \{|V_-^{(t-1)} \cap V_+^{(t)}| \geqslant \delta n^{1/2+\delta}\}\right] = o(1). \tag{15}$$

Let $\mathcal{F}_t$ be the $\sigma$-algebra generated by $(G_s, \mathbf{W}_s)_{0 \leqslant s \leqslant t-1}$, i.e., the first $t - 1$ days of the dynamics. In the following, we condition on $\mathcal{F}_t$ and let $v \in V_-^{(t)}$. Denote by $\mathbb{P}_t$ the corresponding conditional probability. Note that $V_-^{(t)}$ is $\mathcal{F}$-measurable while $v_+^{(t)}$ is not. By definition, on the event $\mathscr{E} \cap \mathcal{N}^c$ we have

$$v_+^{(t-1)} \geqslant v_+^{(0)} \geqslant (n + \Delta)q - n^{(\delta+1)/2} \text{ and } v_-^{(t)} \leqslant v_-^{(0)} \leqslant np + n^{(\delta+1)/2}.$$

On the other hand, on the event $\{|V_-^{(t-1)} \cap V_+^{(t)}| \geqslant \delta n^{1/2+\delta}\} \cap \mathcal{N}^c$, it holds that

$$v_+^{(t)} \geqslant_{\mathrm{st}} v_+^{(t-1)} + \mathsf{Bin}(\delta n^{1/2+\delta}, q) \geqslant (n+\Delta)q - n^{(\delta+1)/2} + \mathsf{Bin}(\delta n^{1/2+\delta}, q),$$

where $\leqslant_{\mathrm{st}}$ means (first-order) stochastic dominance. It follows that on the event $\mathcal{X} := \{|V_-^{(t-1)} \cap V_+^{(t)}| \geqslant \delta n^{1/2+\delta}\} \cap \mathcal{N}^c \cap \mathcal{E}$, we have

$$
\begin{aligned}
\mathbb{P}_t\left[\{v \notin V_+^{(t+1)}\} \cap \mathcal{X}\right] &= \mathbb{P}_t\left[\{v_+^{(t)} \leqslant v_-^{(t)}\} \cap \mathcal{X}\right] \\
&\leqslant \mathbb{P}\left[\{(n+\Delta)q - n^{(\delta+1)/2} + \mathsf{Bin}(\delta n^{1/2+\delta}, q) \leqslant np + n^{(\delta+1)/2}\} \cap \mathcal{X}\right] \\
&\leqslant \exp\left(-2\delta q^2 n^{(\delta+1)/2}\right),
\end{aligned}
$$

where we have used our assumption $\Delta' = o(n^{1/2+\delta})$ and (9) in the last inequality. We finally conclude from a union bound that

$$\mathbb{P}_t\left[\mathscr{P}_{t+1}^c \cap \mathcal{E} \cap \mathcal{N}^c \cap \{|V_-^{(t-1)} \cap V_+^{(t)}| \geqslant \delta n^{1/2+\delta}\}\right] \leqslant n\exp\left(-2\delta q^2 n^{(\delta+1)/2}\right).$$

Taking expectation yields the claim (15) and hence completes the proof. $\square$

**Proposition 4.3.** *Let $\delta > 0$ and $t \in \mathbb{N}_0$. For the Markovian model,*

$$\mathbb{P}[\mathscr{P}_{t+1}] \geqslant \mathbb{P}\left[|V_+^{(t)}| \geqslant \frac{p}{q}|V_-^{(t)}| + L\sqrt{|V_-^{(t)}| \log |V_-^{(t)}|}\right] - o(1).$$

*Proof.* By the Markov property, we may assume $t = 0$ and write $n = |V_-^{(t)}|$ and

$$\Delta(n) \geqslant \left(\frac{p-q}{q}\right)n + L_1\sqrt{n \log n}.$$

Using similar arguments as in Lemma 3.3, one can prove that by choosing $L_1$ above large enough, for any $v \in V_-^{(0)}$,

$$\mathbb{P}\left[v \notin V_-^{(1)}\right] \leqslant Ln^{-2}.$$

Using a union bound over the set $V_-^{(0)}$ finishes the proof. $\square$

Next, we show for the Markovian model that a sufficient lead of $\Omega(n)$ will guarantee a win a.a.s.

**Proposition 4.4.** *Let $\delta > 0$ be arbitrary, then for the Markovian model, uniformly for $n, \Delta$ such that $\Delta > \delta n$, it holds $\mathbb{P}[\mathscr{P}] = 1 - o(1)$.*

*Proof.* Recall that $\{|V_+^{(t)}|\}$ forms a Markovian random walk on $[0, 2n + \Delta]$. We have showed in Proposition 4.2 that once the random walk reaches $2n + \Delta - n/L$ then $\mathscr{P}$ happens with high probability; thus it suffices if we show that starting from $n + \Delta$, the random walk is monotonically non-decreasing. This motivates the study of the following conditional probability. Writing $|V_-^{(0)}| = n$ and $|V_+^{(0)}| = n + \Delta$, We have

$$
\begin{aligned}
p_{n,n+\Delta} &:= \mathbb{P}\left[|V_+^{(0)} \cap V_-^{(1)}| \geqslant 1 \,\Big|\, |V_+^{(0)} \cap V_-^{(1)}| + |V_-^{(0)} \cap V_+^{(1)}| \neq 0\right] \\
&= \frac{\mathbb{P}\left[|V_+^{(0)} \cap V_-^{(1)}| \geqslant 1\right]}{\mathbb{P}\left[|V_+^{(0)} \cap V_-^{(1)}| + |V_-^{(0)} \cap V_+^{(1)}| \neq 0\right]} \leqslant \frac{\mathbb{P}\left[|V_+^{(0)} \cap V_-^{(1)}| \geqslant 1\right]}{\mathbb{P}\left[|V_-^{(0)} \cap V_+^{(1)}| \geqslant 1\right]}.
\end{aligned}
$$

18

Using a union bound and similar arguments as in Lemma 3.3, we compute

$$\mathbb{P}\left[|V_+^{(0)} \cap V_-^{(1)}| \geqslant 1\right] \leqslant Ln\mathbb{P}[\mathsf{Bin}(n,q) > \mathsf{Bin}(n+\Delta-1,p)]$$

$$\leqslant Ln^2 \max_{j\in\mathbb{Z}\cap[nq,(n+\Delta-1)p]} \mathbb{P}[\mathsf{Bin}(n,q) > j]\,\mathbb{P}[\mathsf{Bin}(n+\Delta-1,p) < j]$$

$$\leqslant Ln^2 \exp(-C_1 n)$$

and

$$\mathbb{P}\left[|V_-^{(0)} \cap V_+^{(1)}| \geqslant 1\right] \geqslant \mathbb{P}[\mathsf{Bin}(n+\Delta,q) > \mathsf{Bin}(n-1,p)]$$

$$\geqslant \max_{k\in\mathbb{Z}\cap[nq,(n+\Delta-1)p]} \mathbb{P}[\mathsf{Bin}(n+\Delta,q) > k]\,\mathbb{P}[\mathsf{Bin}(n-1,p) < k]$$

$$\geqslant \frac{1}{L}\exp(-C_2 n)$$

where $C_1, C_2$ are constants that do not depend on $n$ and $C_2 < C_1$ (here we use $\Delta > \delta n$). This shows $p_{n,n+\Delta} \leqslant L\exp(-n/L)$. Using a union bound shows that with probability $1 - o(1)$, the random walk $\{|V_+^{(t)}|\}$ increases monotonically from $n + \Delta$ to $2n + \Delta - n/L$, completing the proof. $\qquad\square$

## 4.2  Proof of Theorem 1

We recall from (8) that $p_{-+} = \mathbb{P}[\mathsf{Bin}(n+\Delta,q) > \mathsf{Bin}(n-1,p)]$. It follows from Hoeffding's inequality that uniformly for $\Delta \in [0, n/L]$, it holds

$$p_{-+} \leqslant L\exp\left(-\frac{n}{L}\right). \tag{16}$$

Next, we study the transition probabilities of the Markov chain $\{|V_+^{(t)}|\}_{t\in\mathbb{N}}$. Consider the Markovian model on $\mathsf{SBM}(j, 2n + \Delta - j, p, q)$ where $j \in [2n + \Delta]$. We denote by

$$p_r(j) = p_r(j; n, \Delta) := \mathbb{P}\left[|V_-^{(0)} \cap V_+^{(1)}| = 1\right] \quad \text{and} \quad p_\ell(j) = p_\ell(j; n, \Delta) := \mathbb{P}\left[|V_+^{(0)} \cap V_-^{(1)}| = 1\right] \tag{17}$$

where $|V_+^{(0)}| = j$ and $|V_-^{(0)}| = 2n + \Delta - j$. As a special case that corresponds to the first step of the Markovian model on $\mathsf{SBM}(n + \Delta, n, p, q)$, we write

$$p_r = p_r(n + \Delta) = \mathbb{P}\left[|V_-^{(0)} \cap V_+^{(1)}| = 1\right] \quad \text{and} \quad p_\ell = p_\ell(n + \Delta) = \mathbb{P}\left[|V_+^{(0)} \cap V_-^{(1)}| = 1\right].$$

**Lemma 4.1.** *Consider the Markovian model on $\mathsf{SBM}(n+\Delta, n, p, q)$ where $n+\Delta = |V_+^{(0)}| > |V_-^{(0)}| = n$. Then uniformly in $n$ and $\Delta \leqslant n/L$, it holds that*

$$\frac{p_r}{p_\ell} \geqslant 1 + \frac{1}{L}, \tag{18}$$

*and*

$$\frac{\mathbb{P}\left[|V_-^{(0)} \cap V_+^{(1)}| \geqslant 2\right] + \mathbb{P}\left[|V_+^{(0)} \cap V_-^{(1)}| \geqslant 2\right]}{p_r + p_\ell} \leqslant L\exp\left(-\frac{n}{L}\right). \tag{19}$$

*Proof.* We label the vertices in $V_+^{(0)}$ by $i \in [n + \Delta]$ and vertices in $V_-^{(0)}$ by $j \in [n]$. Recall that the degree sequences of the graph $\mathsf{SBM}(n + \Delta, n, p, q)$ are sampled from three independent random graphs $\mathsf{G}(n + \Delta, p)$, $\mathsf{G}(n, p)$ and $\mathsf{G}(n + \Delta, n, q)$, and we denote by $\mathbf{D}_+$, $\mathbf{D}_-$, and $(\mathbf{S}, \mathbf{T})$ the degree sequences of these subgraphs, respectively.

Note that

$$\mathbb{P}\left[|V_-^{(0)} \cap V_+^{(1)}| = 1\right] \leqslant \sum_{j=1}^{n} \mathbb{P}\left[j \in V_-^{(0)} \cap V_+^{(1)}\right] = n\,\mathbb{P}\left[\mathbf{T}(1) > \mathbf{D}_-(1)\right]$$

$$= n\,\mathbb{P}\left[\mathsf{Bin}(n + \Delta, q) > \mathsf{Bin}(n - 1, p)\right] = np_{-+},$$

where $p_{-+}$ is defined in (8). On the other hand, by the union bound,

$$\mathbb{P}\left[|V_-^{(0)} \cap V_+^{(1)}| = 1\right] = \sum_{j=1}^{n} \mathbb{P}\left[j \in V_-^{(0)} \cap V_+^{(1)}, \text{ and } k \notin V_-^{(0)} \cap V_+^{(1)} \text{ for all } k \neq j\right]$$

$$\geqslant \sum_{j=1}^{n} \left(\mathbb{P}\left[j \in V_-^{(0)} \cap V_+^{(1)}\right] - \sum_{\substack{1 \leqslant k \leqslant n \\ k \neq j}} \mathbb{P}\left[j, k \in V_-^{(0)} \cap V_+^{(1)}\right]\right)$$

$$= \sum_{j=1}^{n} \mathbb{P}\left[j \in V_-^{(0)} \cap V_+^{(1)}\right] - \sum_{\substack{1 \leqslant j,k \leqslant n \\ k \neq j}} \mathbb{P}\left[j, k \in V_-^{(0)} \cap V_+^{(1)}\right].$$

To estimate the second term, we observe that for $j \neq k$,

$$\mathbb{P}\left[j, k \in V_-^{(0)} \cap V_+^{(1)}\right] \leqslant \widetilde{\mathbb{P}}\left[j, k \in V_-^{(0)} \cap V_+^{(1)}\right]$$

where $\widetilde{\mathbb{P}}$ is the probability measure of the graph $G \sim \mathsf{SBM}(n + \Delta, n, p, q)$ conditioned on $(j, k) \notin E$. Let $\widetilde{\mathbf{D}}_+$, $\widetilde{\mathbf{D}}_-$, and $(\widetilde{\mathbf{S}}, \widetilde{\mathbf{T}})$ denote the degree sequences with respect to $\widetilde{\mathbb{P}}$. Then,

$$\widetilde{\mathbb{P}}\left[j, k \in V_-^{(0)} \cap V_+^{(1)}\right] \leqslant \widetilde{\mathbb{P}}\left[\widetilde{\mathbf{T}}(j) > \widetilde{\mathbf{D}}_-(j)\right] \widetilde{\mathbb{P}}\left[\widetilde{\mathbf{T}}(k) > \widetilde{\mathbf{D}}_-(k)\right]$$

$$\leqslant \left(\mathbb{P}\left[\mathsf{Bin}(n + \Delta, q) > \mathsf{Bin}(n - 2, p)\right]\right)^2$$

$$\leqslant L\left(\mathbb{P}\left[\mathsf{Bin}(n + \Delta, q) > \mathsf{Bin}(n - 1, p)\right]\right)^2$$

$$\leqslant Lp_{-+}^2.$$

This implies that

$$\mathbb{P}\left[|V_-^{(0)} \cap V_+^{(1)}| = 1\right] \geqslant np_{-+} - Ln^2 p_{-+}^2 = (1 - o(1))np_{-+},$$

where the last step follows from (16). Therefore, we conclude that

$$p_r = \mathbb{P}\left[|V_-^{(0)} \cap V_+^{(1)}| = 1\right] = (1 - o(1))np_{-+},$$

and a similar argument yields

$$p_\ell = (1 - o(1))(n + \Delta)p_{+-}.$$

It follows that

$$\frac{p_r}{p_\ell} \geqslant (1 - o(1))\frac{np_{-+}}{(n + \Delta)p_{+-}} \geqslant (1 - o(1))\left(1 - \frac{1}{L_2 + 1}\right)\frac{p_{-+}}{p_{+-}}$$

20

where we used $\Delta \leqslant n/L_2$ with $L_2$ a large constant to be determined.

Consider the independent random variables $T \sim \mathsf{Bin}(n+\Delta, q)$, $S \sim \mathsf{Bin}(n, q)$, $D_- \sim \mathsf{Bin}(n-1, p)$ and $D_+ \sim \mathsf{Bin}(n+\Delta-1, p)$. By choosing $L_2$ above large enough, in order to prove (18), it suffices to show

$$\frac{\mathbb{P}[T > D_-]}{\mathbb{P}[S > D_+]} \geqslant 1 + \frac{1}{L}. \tag{20}$$

Since $\mathsf{Bin}(n-1, p) \leqslant_{\mathrm{st}} \mathsf{Bin}(n+\Delta-1, p)$, we may replace $D_-$ in the numerator by $D_+$. Similarly, using stochastic dominance again, we may replace $T$ with $T' \sim \mathsf{Bin}(n+1, q)$. By a suitable coupling, we may assume $T' = S + \epsilon$ where $\epsilon \sim \mathsf{Bin}(1, q)$ is independent from $S$, and $(S, T')$ and $D_+$ are independent. It follows that

$$\frac{\mathbb{P}[T > D_-]}{\mathbb{P}[S > D_+]} - 1 \geqslant \frac{\mathbb{P}[T > D_+]}{\mathbb{P}[S > D_+]} - 1 \geqslant \frac{\mathbb{P}[T' > D_+]}{\mathbb{P}[S > D_+]} - 1 = \frac{\mathbb{P}[S = D_+, \epsilon = 1]}{\mathbb{P}[S > D_+]} = \frac{q\,\mathbb{P}[S = D_+]}{\mathbb{P}[S > D_+]}. \tag{21}$$

To estimate the right-hand side of (21), note that

$$\frac{\mathbb{P}[S = D_+]}{\mathbb{P}[S = D_+ + 1]} = \frac{\sum_{x=0}^{n+\Delta-1} \mathbb{P}[D_+ = x]\,\mathbb{P}[S = x]}{\sum_{x=0}^{n+\Delta-1} \mathbb{P}[D_+ = x]\,\mathbb{P}[S = x+1]}.$$

Using log-concavity of the functions $x \mapsto \mathbb{P}[D_+ = x]\,\mathbb{P}[S = x]$ and $x \mapsto \mathbb{P}[D_+ = x]\,\mathbb{P}[S = x+1]$ and that $\Delta \leqslant n/L_2$, it is not hard to show that there exist $q_1, q_2$ depending only on $p, q$ satisfying $nq < nq_1 < nq_2 < (n+\Delta)p$ so that

$$\frac{\sum_{x=0}^{n+\Delta-1} \mathbb{P}[D_+ = x]\,\mathbb{P}[S = x]}{\sum_{x=0}^{n+\Delta-1} \mathbb{P}[D_+ = x]\,\mathbb{P}[S = x+1]} = (1 + o(1))\frac{\sum_{x=nq_1}^{nq_2} \mathbb{P}[D_+ = x]\,\mathbb{P}[S = x]}{\sum_{x=nq_1}^{nq_2} \mathbb{P}[D_+ = x]\,\mathbb{P}[S = x+1]},$$

and for some $L_3 > 0$,

$$\frac{\mathbb{P}[S = x]}{\mathbb{P}[S = x+1]} = (1 + o(1))\frac{(1-q)x}{q(n-x)} \geqslant 1 + \frac{1}{L_3}, \quad \text{for } x \in [nq_1, nq_2] \cap \mathbb{Z}.$$

This implies that

$$\frac{\mathbb{P}[S = D_+]}{\mathbb{P}[S = D_+ + 1]} \geqslant 1 + \frac{1}{L_3}. \tag{22}$$

Recall that the probability mass function of a binomial distribution is log-concave, and that a convolution of log-concave functions is log-concave. This implies the probability mass function of $S - D_+$ is log-concave. Combined with (22), we obtain

$$\mathbb{P}[S > D_+] \leqslant \sum_{x=1}^{n+\Delta-1} \mathbb{P}[S - D_+ = x] \leqslant \sum_{x=1}^{n+\Delta-1} \left(\frac{L_3}{L_3+1}\right)^{x-1} \mathbb{P}[S = D_+ + 1]$$
$$\leqslant (L_3 + 1)\mathbb{P}[S = D_+ + 1].$$

Consequently, we have

$$\frac{\mathbb{P}[S = D_+]}{\mathbb{P}[S > D_+]} \geqslant \frac{\mathbb{P}[S = D_+]}{(L_3 + 1)\mathbb{P}[S = D_+ + 1]} \geqslant \frac{1}{L}.$$

Plugging back into (21) yields the desired (20).

To show (19), by the union bound, for $i \neq j$,

$$\mathbb{P}\left[|V_-^{(0)} \cap V_+^{(1)}| \geqslant 2\right] \leqslant n^2\,\mathbb{P}\left[i, j \in V_-^{(0)} \cap V_+^{(1)}\right],$$

and recall that for $i \neq j$,

$$\mathbb{P}\left[i, j \in V_-^{(0)} \cap V_+^{(1)}\right] \leqslant L p_{-+}^2.$$

Therefore, it holds that

$$\frac{\mathbb{P}\left[|V_-^{(0)} \cap V_+^{(1)}| \geqslant 2\right]}{p_r} \leqslant \frac{L n^2 p_{-+}^2}{(1 - o(1)) n p_{-+}} \leqslant L \exp\left(-\frac{n}{L}\right),$$

where the last step follows from (16). Similarly, we also have

$$\frac{\mathbb{P}\left[|V_+^{(0)} \cap V_-^{(1)}| \geqslant 2\right]}{p_\ell} \leqslant L \exp\left(-\frac{n}{L}\right).$$

Combining these together yields the desired result. $\square$

*Proof of Theorem 1(i).* We fix $p, q \in [0, 1]$ with $0 \leqslant q < p \leqslant 1$ and $\Delta = 1$, where the general case $\Delta \geqslant 1$ can be established using exactly the same proof with the same constant $L$. In this case, $\{|V_+^{(t)}|\}$ is a Markovian random walk on $[0, 2n + 1] \cap \mathbb{Z}$ whose transition probabilities are symmetric along $n + 1/2$.

Recall (17). By Lemma 4.1, there exists $L_2$ such that the probabilities $p_r(j)$ and $p_\ell(j)$ at point $j \in [n + 1, n + n/L_2] \cap \mathbb{Z}$ satisfy $p_r(j)/p_\ell(j) \geqslant 1 + 1/L$ uniformly. It follows from Proposition 4.4 that given the random walk $\{|V_+^{(t)}|\}$ reaches $n + n/L_2$, $\mathscr{P}$ happens with probability $1 - o(1)$. Thus by Markov property and symmetry of the random walk, it suffices to prove that $\{|V_+^{(t)}|\}$ reaches $n + n/L_2$ before $n$ with probability at least $1/L$. Define $\tau$ the hitting time of the random walk with boundaries $n + n/L_2$ and $n$. Note that for $t < \tau$ on the event $A_t := \{|V_-^{(t)} \cap V_+^{(t+1)}| \leqslant 1$ and $|V_+^{(t)} \cap V_-^{(t+1)}| \leqslant 1\}$,

$$\frac{\mathbb{P}\left[|V_+^{(t+1)}| - |V_+^{(t)}| = 1\right]}{\mathbb{P}\left[|V_+^{(t+1)}| - |V_+^{(t)}| = -1\right]} = \frac{\mathbb{P}\left[|V_-^{(t)} \cap V_+^{(t+1)}| = 1 \text{ and } |V_+^{(t)} \cap V_-^{(t+1)}| = 0\right]}{\mathbb{P}\left[|V_-^{(t)} \cap V_+^{(t+1)}| = 0 \text{ and } |V_+^{(t)} \cap V_-^{(t+1)}| = 1\right]} \geqslant \frac{p_r(|V_+^{(t)}|)}{p_\ell(|V_+^{(t)}|)} \geqslant 1 + \frac{1}{L}.$$

We say that the random walk performs a non-trivial move at time $t$ if $|V_+^{(t+1)}| \neq |V_+^{(t)}|$. Define the event

$$A = \bigcap_{\left\{t < \tau : |V_+^{(t)}| \neq |V_+^{(t+1)}|\right\}} A_t.$$

This means that before the random walk is stopped, whenever it moves, the step length is one. Taking intersection with the event $A$, it follows from a Gambler's Ruin argument (e.g., [CCH00, Theorem 1]) that $\mathbb{P}[|V_+^{(\tau)}| = n + n/L_2] \geqslant 1/L$. It then suffices to prove that $\mathbb{P}[A] = 1 - o(1)$. By Lemma 4.1 and union bound, with probability $1 - o(1)$, each step length of the first $n^3$ non-trivial moves of $\{|V_+^{(t)}|\}$ is one. Intersecting on this event, the probability that $\tau < n^3$, hence $A$ occurs, is $1 - o(1)$ due to standard estimates on hitting times (e.g., [Fel08]). This completes the proof of (i).

Part (ii). Assume $\Delta(n) \to \infty$ and pick another sequence $\kappa(n) \to \infty$ such that $\kappa(n) = o(\Delta(n))$. Let $L_2$ be given as in Lemma 4.1. By Proposition 4.4, it suffices to show that the Markovian random walk $\{|V_+^{(t)}|\}$ reaches $n + \Delta + n/L_2$ before $n + \Delta - \kappa(n)$ with probability tending to one. Using similar arguments as in the proof of (i), we may intersect with the event that when the

22

random walk moves, it is either the case $|V_-^{(t)} \cap V_+^{(t+1)}| = 1$ and $|V_+^{(t)} \cap V_-^{(t+1)}| = 0$, or the case $|V_-^{(t)} \cap V_+^{(t+1)}| = 0$ and $|V_+^{(t)} \cap V_-^{(t+1)}| = 1$. Since

$$\frac{\mathbb{P}\left[|V_-^{(t)} \cap V_+^{(t+1)}| = 1 \text{ and } |V_+^{(t)} \cap V_-^{(t+1)}| = 0\right]}{\mathbb{P}\left[|V_-^{(t)} \cap V_+^{(t+1)}| = 0 \text{ and } |V_+^{(t)} \cap V_-^{(t+1)}| = 1\right]} \geqslant 1 + \frac{1}{L}$$

uniformly given $n + \Delta - \kappa(n) \leqslant |V_+^{(t)}| \leqslant n + \Delta + n/L_2$, a standard Gambler's Ruin estimate (e.g., [CCH00, Theorem 1]) completes the proof. $\qquad\square$

## 4.3   Proof of Theorems 2 and 3

By Proposition 4.1, it would be convenient for us to exclude the event $\mathscr{N}$ and assume that there is no + opinion turning $-$. We will implicitly intersect with the event $\mathscr{N}^c$ throughout the proofs below.

*Proof of Theorem 2.* (i). Note that the event $\mathscr{T}$ contains the event that the dynamics halts at day $t$ with $0 < |V_+^{(t)}| < 2n + \Delta$, i.e.,

$$\mathbb{P}[\mathscr{T}] \geqslant \mathbb{P}\left[|V_-^{(t)} \cap V_+^{(t+1)}| = |V_+^{(t)} \cap V_-^{(t+1)}| = 0 \text{ and } 0 < |V_+^{(t)}| < 2n + \Delta\right]. \tag{23}$$

Taking $t = 0$ in (23), this yields a first trivial lower bound

$$\mathbb{P}[\mathscr{T}] \geqslant \mathbb{P}\left[|V_-^{(0)} \cap V_+^{(1)}| = |V_+^{(0)} \cap V_-^{(1)}| = 0\right].$$

We then use a union bound to control the right-hand side from below. Recall (8). Assume now that

$$\Delta(n) \leqslant \left(\frac{p-q}{q}\right)n - \sqrt{\frac{n(\log n + \log\log n + \omega(1))}{C'}}, \tag{24}$$

where $C'$ is given by (14). Lemma 3.3 then gives that $p_{-+} = o(n^{-1})$. On the other hand, (9) yields

$$p_{+-} \leqslant \mathbb{P}\left[\mathsf{Bin}(n, q) \geqslant \left(\frac{p+q}{2}\right)n\right] + \mathbb{P}\left[\mathsf{Bin}(n + \Delta - 1, p) \leqslant \left(\frac{p+q}{2}\right)n\right] \leqslant L\exp\left(-\frac{n}{L}\right).$$

Using a union bound, we compute

$$\begin{aligned}
\mathbb{P}[\mathscr{T}] &\geqslant \mathbb{P}\left[|V_-^{(0)} \cap V_+^{(1)}| = |V_+^{(0)} \cap V_-^{(1)}| = 0\right] \\
&\geqslant 1 - \sum_{v \in V_+^{(0)}} \mathbb{P}\left[v \in V_+^{(0)} \cap V_-^{(1)}\right] - \sum_{w \in V_-^{(0)}} \mathbb{P}\left[w \in V_-^{(0)} \cap V_+^{(1)}\right] \\
&= 1 - (n + \Delta)p_{+-} - np_{-+} \geqslant 1 - o(1).
\end{aligned}$$

We may assume now that (24) does not hold, so that Lemma 3.4 applies. To improve the bound in (24) we let $t = 1$ in (23) and consider the lower bound

$$\mathbb{P}[\mathscr{T}] \geqslant \mathbb{P}\left[|V_-^{(1)} \cap V_+^{(2)}| = |V_+^{(1)} \cap V_-^{(2)}| = 0 \text{ and } |V_-^{(0)} \cap V_+^{(1)}| < n + \Delta\right],$$

23

that is, the probability that the dynamics halts after day two. It follows from Proposition 4.1 that $|V_+^{(1)} \cap V_-^{(2)}| = 0$ a.a.s., so we bound $\mathbb{P}[|V_-^{(1)} \cap V_+^{(2)}| = 0]$ from below. Assuming (2), we have by Lemma 3.1 that with $a_n = L(n/\log n)^{1/4}/\exp(\sqrt{d_n})$,

$$
\begin{aligned}
\mathbb{P}\left[|V_-^{(0)} \cap V_+^{(1)}| \leqslant a_n\right] &= (1 - o(1))\mathbb{P}_{\mathcal{B}_{r_2}^n, \mathcal{B}_q^{n+\Delta, n}}\left[|V_-^{(0)} \cap V_+^{(1)}| \leqslant a_n\right] - o(1) \\
&= (1 - o(1))\mathbb{P}[\mathsf{Bin}(n, p_{-+}) \leqslant a_n] - o(1) \\
&\geqslant (1 - o(1))\mathbb{P}\left[\mathsf{Bin}\left(n, \frac{Ln^{-3/4}(\log n)^{-1/4}}{\exp(\frac{\sqrt{6d_n}}{2})}\right) \leqslant a_n\right] - o(1) \\
&= 1 - o(1),
\end{aligned}
$$

where we used Lemma 3.3 in the inequality. Thus we may intersect with the event $\{|V_-^{(0)} \cap V_+^{(1)}| \leqslant a_n\}$ while removing a set of probability $o(1)$.

Consider a fixed vertex $v \in V_-^{(0)}$. Our goal is to bound $\mathbb{P}[v \in V_-^{(0)} \cap V_-^{(1)} \cap V_+^{(2)}]$ from above. Denote for $j = 0, 1$ by $v_+^{(j)}, v_-^{(j)}$ the number of positive and negative neighbors of $v$ at day $j$. Note that

(a) $v_+^{(1)} \leqslant v_+^{(0)} + \mathsf{Bin}(a_n, q)$, so that $v_+^{(1)} \leqslant v_+^{(0)} + a_n$ a.a.s.;

(b) $v_-^{(1)} \geqslant v_-^{(0)} - a_n$;

(c) $v_+^{(0)} \sim \mathsf{Bin}(n + \Delta, q)$ and $v_-^{(0)} \sim \mathsf{Bin}(n - 1, r_2)$ are independent.

Therefore, by Lemma 3.4,

$$
\begin{aligned}
\mathbb{P}\left[v \in V_-^{(0)} \cap V_-^{(1)} \cap V_+^{(2)}\right] &= \mathbb{P}\left[v_-^{(0)} \geqslant v_+^{(0)} \text{ and } v_-^{(1)} < v_+^{(1)}\right] \\
&\leqslant \mathbb{P}\left[v_-^{(0)} \geqslant v_+^{(0)} \text{ and } v_-^{(0)} \leqslant v_+^{(0)} + a_n\right] \\
&= \mathbb{P}[\mathsf{Bin}(n + \Delta, q) \leqslant \mathsf{Bin}(n - 1, r_2) \leqslant \mathsf{Bin}(n + \Delta, q) + 2a_n] \\
&= o(n^{-1}).
\end{aligned}
$$

Thus it follows from the union bound that $V_-^{(0)} \cap V_-^{(1)} \cap V_+^{(2)} = \emptyset$ a.a.s., proving (i).

Part (ii). This follows directly from Proposition 4.3, since the first step is the same for both Markovian and non-Markovian models.

Part (iii). We may assume by (ii) that

$$
\Delta(n) \leqslant \left(\frac{p - q}{q}\right)n + L\sqrt{n \log n},
$$

so that Lemma 3.3 applies. By Proposition 4.2, it suffices to prove that for some $\delta > 0$, $\mathbb{P}[|V_-^{(0)} \cap V_+^{(1)}| \leqslant \delta n^{(\delta+1)/2}] = o(1)$. By Lemma 3.1, it suffices to show for some $\delta > 0$ that for all $r_2 \in [p - L(\log n)/n, p + L(\log n)/n]$,

$$
\mathbb{P}_{\mathcal{B}_{r_2}^n, \mathcal{B}_q^{n+\Delta, n}}\left[|V_-^{(0)} \cap V_+^{(1)}| \leqslant \delta n^{(\delta+1)/2}\right] = o(1).
$$

Recall our assumption that for some $\delta_0 > 0$,

$$
\Delta' = \frac{(p - q)n}{q} - \Delta \leqslant \sqrt{\frac{(1/2 - \delta_0)n \log n}{C'}}.
$$

Together with Lemma 3.3, this yields that

$$\mathbb{P}[\mathsf{Bin}(n-1, r_2) < \mathsf{Bin}(n+\Delta, q)] \geqslant \frac{1}{L} n^{(\delta_0+1)/2}.$$

Thus by picking $\delta$ small enough, we have

$$\mathbb{P}_{\mathcal{B}^n_{r_2}, \mathcal{B}^{n+\Delta,n}_q} \left[ |V^{(0)}_- \cap V^{(1)}_+| \leqslant \delta n^{(\delta+1)/2} \right]$$

$$= \mathbb{P}\left[ \mathsf{Bin}\left(n, \mathbb{P}[\mathsf{Bin}(n-1, r_2) < \mathsf{Bin}(n+\Delta, q)]\right) \leqslant \delta n^{(\delta+1)/2} \right] = o(1).$$

This concludes the proof.

Part (iv). Again by Proposition 4.2, it suffices to prove $\mathbb{P}[|V^{(1)}_- \cap V^{(2)}_+| \leqslant n^{(\delta+1)/2}] = o(1)$. Let $r_2 \in [p - L(\log n)/n, p + L(\log n)/n]$ be arbitrary. Using (iii), we may assume that

$$\Delta(n) \leqslant \left(\frac{p-q}{q}\right) n - \sqrt{\frac{(1/2 - \delta)n \log n}{C'}},$$

so that, by Lemma 3.3, it holds $\widetilde{p}_{-+} \leqslant Ln^{-2\gamma}$ for some $\gamma > 0$. By Markov's inequality, we have

$$\mathbb{P}_{\mathcal{B}^n_{r_2}, \mathcal{B}^{n+\Delta,n}_q} \left[ \sum_{j=1}^n \mathbb{1}_{\{\mathbf{D}_2(j) \geqslant \mathbf{T}(j)\}} \geqslant n^{1-\gamma} \right] = o(1). \tag{25}$$

Next, we show that

$$\mathbb{P}\left[ |V^{(0)}_- \cap V^{(1)}_+| \geqslant L_4 \sqrt{n \log n} \right] = 1 - o(1) \tag{26}$$

where $L_4$ is a large constant to be determined. This can be proved in a similar way as in (iii) as follows. It follows from our assumption (3) (with the constant $L$ in (3) chosen large enough depending on $L_4$) and Lemma 3.3 that

$$\mathbb{P}_{\mathcal{B}^n_{r_2}, \mathcal{B}^{n+\Delta,n}_q} \left[ |V^{(0)}_- \cap V^{(1)}_+| \leqslant L_4 \sqrt{n \log n} \right]$$

$$= \mathbb{P}\left[ \mathsf{Bin}\left(n, \mathbb{P}[\mathsf{Bin}(n-1, r_2) < \mathsf{Bin}(n+\Delta, q)]\right) \leqslant L_4 \sqrt{n \log n} \right]$$

$$\leqslant \mathbb{P}\left[ \mathsf{Bin}\left(n, 2L_4 \sqrt{\frac{\log n}{n}}\right) \leqslant L_4 \sqrt{n \log n} \right] = o(1).$$

Let us continue to the proof of $\mathbb{P}[|V^{(1)}_- \cap V^{(2)}_+| \leqslant n/2] = o(1)$, which is stronger than what we need. We may intersect with the event $\{|V^{(0)}_- \cap V^{(1)}_+| \geqslant L_4 \sqrt{n \log n}\}$, thus removing a set of probability $o(1)$ by (26). Using Lemma 3.1, we may also move the probability measure from $\mathbb{P}$ to the i.i.d. model $\mathbb{P}_{\mathcal{B}^n_{r_2}, \mathcal{B}^{n+\Delta,n}_q}$ where $|r_2 - p| \leqslant L(\log n)/n$. Fix such an $r_2$. We condition on the $\sigma$-algebra generated by the initial configuration of the SBM, i.e., $\mathbf{D}_1, \mathbf{D}_2, \mathbf{S}, \mathbf{T}$, so that the remaining randomness are the edge changes induced by the opinion changes after day one. It holds that

$$|V^{(1)}_- \cap V^{(2)}_+| \geqslant |V^{(0)}_- \cap V^{(1)}_- \cap V^{(2)}_+| \geqslant \sum_{j=1}^n \mathbb{1}_{\{0 < \mathbf{D}_2(j) - \mathbf{T}(j) < \xi_j\}}, \tag{27}$$

25

where $\xi_j \sim \mathsf{Bin}(L_4\sqrt{n\log n}, q)$ is an i.i.d. sequence independent of everything else. This is because for a vertex $j \in V_-^{(0)}$, if $\mathbf{D}_2(j) - \mathbf{T}(j) > 0$ then $j \in V_-^{(1)}$, and if $\mathbf{D}_2(j) - \mathbf{T}(j) < \xi_j$ (where $\xi_j$ represents the new edges connecting $j$ to $V_-^{(0)} \cap V_+^{(1)}$, which has a contribution with distribution $\mathsf{Bin}(|V_-^{(0)} \cap V_+^{(1)}|, q)$ conditioned on $V_-^{(0)} \cap V_+^{(1)}$ by similar arguments as in Proposition 4.2), then $x \in V_+^{(2)}$ (on the event $\mathcal{N}^c$). We obtain from (27) that

$$\mathbb{P}\left[|V_-^{(1)} \cap V_+^{(2)}| > \frac{n}{2}\right] = (1 + o(1))\mathbb{P}_{\mathcal{B}_{r_2}^n, \mathcal{B}_q^{n+\Delta,n}}\left[|V_-^{(1)} \cap V_+^{(2)}| > \frac{n}{2}\right] - o(1)$$

$$\geqslant (1 + o(1))\mathbb{P}_{\mathcal{B}_{r_2}^n, \mathcal{B}_q^{n+\Delta,n}}\left[\sum_{j=1}^n \mathbb{1}_{\{0 < \mathbf{D}_2(j) - \mathbf{T}(j) < \xi_j\}} > \frac{n}{2}\right] - o(1).$$

To evaluate the probability above, we condition on everything except for $\{\xi_j\}$, which is i.i.d. in $j$ and independent of everything else. Since $\xi_j \sim \mathsf{Bin}(L_4\sqrt{n\log n}, q)$, it suffices to prove (using that we can pick $L_4$ large enough) for some $L_5 > 0$ that

$$\mathbb{P}_{\mathcal{B}_{r_2}^n, \mathcal{B}_q^{n+\Delta,n}}\left[\sum_{j=1}^n \mathbb{1}_{\{0 < \mathbf{D}_2(j) - \mathbf{T}(j) < L_5\sqrt{n\log n}\}} > \frac{n}{2}\right] \geqslant 1 - o(1).$$

By (25) and definition, this reduces to proving

$$\mathbb{P}\left[\mathsf{Bin}(n, \mathbb{P}[\mathsf{Bin}(n-1, r_2) - \mathsf{Bin}(n+\Delta, q) < L_5\sqrt{n\log n}]) > \frac{2n}{3}\right] \geqslant 1 - o(1).$$

Since $\Delta' \leqslant L\sqrt{n\log n}$, the probability inside is $1 - o(1)$ if $L_5$ is chosen large enough. This completes the proof.

Part (v) is a direct consequence of Proposition 4.1. □

*Proof of Theorem 3.* (i) Similarly as in the first part of the proof of Theorem 2(i), it suffices to consider the case

$$\Delta \geqslant \left(\frac{1-q}{q}\right)n - L\sqrt{n\log n},$$

so that Lemma 3.2 applies. Suppose that

$$\Delta(n) \leqslant \left(\frac{1-q}{q}\right)n - \frac{\sqrt{1-q}}{q}\left(\sqrt{n\log n} + L_6\sqrt{\frac{n(\log\log n)^2}{\log n}}\right),$$

where $L_6$ is a large constant to be determined. We may intersect with $\mathcal{N}^c$ by Proposition 4.1.[1] The key is to note that, conditioning on the sets $V_-^{(0)}, \ldots, V_-^{(t)}$, the number of changes from $-$ to $+$ at time $t$, $|V_-^{(t)} \cap V_+^{(t+1)}|$, follows the binomial distribution $\mathsf{Bin}(|V_-^{(t)}|, p_t)$, where

$$p_t := p_t(V_-^{(0)}, \ldots, V_-^{(t)})$$

$$= \mathbb{P}\left[|V_-^{(t)}| - \mathsf{Bin}(|V_-^{(0)} \cap V_+^{(t-1)}|, q) - \mathsf{Bin}(|V_-^{(t-1)} \cap V_+^{(t)}|, q)\right.$$

$$\left. \leqslant \mathsf{Bin}(n+\Delta, q) \leqslant |V_-^{(t)}| - 1 - \mathsf{Bin}(|V_-^{(0)} \cap V_+^{(t-1)}|, q)\right]. \quad (28)$$

---

[1]Formally, we replace our model by one that does not allow opinion changing from $+$ to $-$.

This follows from the conditional independence of the number of $+$ and $-$ connections of each vertex in $V_-^{(t)}$. Indeed, since $\{V_-^{(t)}\}$ is a decreasing set in $t$, we have for $v \in V_-^{(t)}$, the number of $+$ neighbors of $v$ is $\mathsf{Bin}(n + \Delta + |V_-^{(0)} \cap V_+^{(t)}|, q)$ and the number of $-$ neighbors is $|V_-^{(t)}| - 1$.

In what follows, $L$ might depend on $\delta$ and w.h.p. means happening with probability $1 - n^{-\omega(1)}$. We claim that with $L_6$ chosen large enough, w.h.p. the following holds for all $t \geqslant 1$:

(a) $|V_-^{(0)} \cap V_+^{(t-1)}| \leqslant \sum_{s=1}^{t-1} \frac{\sqrt{n}}{(\log n)^{2s}} \leqslant \frac{L\sqrt{n}}{(\log n)^2}$;

(b) $|V_-^{(t-1)} \cap V_+^{(t)}| \leqslant \frac{\sqrt{n}}{(\log n)^{2t}}$.

Given this claim, the proof is immediate, since this yields that the dynamics is stopped at time $t = \log n$ w.h.p., while the number of opinion changes from $-$ to $+$ until time $t = \log n$ is bounded by $\frac{L\sqrt{n}}{(\log n)^2} = o(n)$, hence the dynamics halts. The total probability of exceptional sets is bounded by $n^{-\omega(1)} \log n = o(1)$.

To prove the above claim we use induction on $t$. The case $t = 1$ follows from a standard computation using our assumption (4) and Lemma 3.2. Given the claim holds for $t$, we have

$$|V_-^{(0)} \cap V_+^{(t)}| \leqslant |V_-^{(0)} \cap V_+^{(t-1)}| + |V_-^{(t-1)} \cap V_+^{(t)}| \leqslant \sum_{s=1}^{t} \frac{\sqrt{n}}{(\log n)^{2s}} \leqslant \frac{L\sqrt{n}}{(\log n)^2}.$$

Moreover, since $|V_-^{(t)}| \leqslant n$, it holds that w.h.p.

$$|V_-^{(t)} \cap V_+^{(t+1)}| \leqslant (\log n)^2 |V_-^{(t)}| p_t \leqslant n (\log n)^2 p_t. \tag{29}$$

By intersecting with the events $|V_-^{(0)} \cap V_+^{(t-1)}| \leqslant \frac{L\sqrt{n}}{(\log n)^2}$ and $|V_-^{(t-1)} \cap V_+^{(t)}| \leqslant \frac{\sqrt{n}}{(\log n)^{2t}}$ in (28), we have

$$p_t \leqslant \mathbb{P}\left[ |V_-^{(t)}| - \mathsf{Bin}(|V_-^{(0)} \cap V_+^{(t-1)}|, q) - \frac{\sqrt{n}}{(\log n)^{2t}} \leqslant \mathsf{Bin}(n + \Delta, q) \right.$$

$$\left. \leqslant |V_-^{(t)}| - \mathsf{Bin}(|V_-^{(0)} \cap V_+^{(t-1)}|, q) \right] + n^{-\omega(1)}$$

$$\leqslant \mathbb{P}\left[ n - \frac{L\sqrt{n}}{(\log n)^2} - \frac{\sqrt{n}}{(\log n)^{2t}} \leqslant \mathsf{Bin}(n + \Delta, q) \leqslant n - \frac{L\sqrt{n}}{(\log n)^2} \right] + n^{-\omega(1)}$$

$$\leqslant \frac{(\log n)^{L_7 - 2t - L_6 + 1}}{\sqrt{n}},$$

where the last step follows from Lemma 3.5 with some appropriate constant $L_7 > 0$. Using (29) we have $|V_-^{(t)} \cap V_+^{(t+1)}| \leqslant \sqrt{n}(\log n)^{L_7 + 3 - 2t - L_6}$. Picking $L_6 \geqslant L_7 + 5$ finishes the induction step.

Parts (ii) and (iii) can be proved similarly as in parts (iii) and (iv) of Theorem 2, applying Lemma 3.2 instead of Lemma 3.3. We leave the details to the reader. $\square$

## 5　Numerical Experiments

In this section, we present some numerical experiments to validate our main theorems and demonstrate the plausibility of Conjecture 1. Throughout this section, we simulate the dynamics for 1000 times and check the frequencies of various outcomes. Recall that $n$ represents the number of vertices with opinion $-$ in the initialization, and $\Delta$ is the initial bias.

## 5.1 Markovian model

For the Markovian model, we first focus on the power-of-one phenomenon. In Table 1, we experiment on the Markovian model with $\Delta = 1, p = 0.5, q = 0.3$ and let the initial number of vertices with opinion $-$ increase.

| Number of initial "−" | 50 | 100 | 125 | 150 | 175 | 200 | 225 | 250 |
|---|---|---|---|---|---|---|---|---|
| Winner | "+" | "+" | "+" | "+" | "+" | "+" | "+" | "+" |
| Averaged last day | 7.68 | 26.90 | 62.21 | 146.36 | 380.47 | 1025.91 | 2757.31 | 6152.87 |
| Frequency | 717 | 677 | 679 | 679 | 687 | 625 | 656 | 633 |

Table 1: Simulation of the Markovian model when $\Delta = 1, p = 0.5, q = 0.3$.

As shown in Table 1, the frequency of the opinion + winning is greater than half of all instances uniformly. This agrees with Part (i) in Theorem 1, that is, a single initial bias already leads to a non-trivial advantage for winning in the end.

Next, recall that Part (ii) of Theorem 1 states that any initial bias of $\omega(1)$ will guarantee an asymptotically almost sure win. To verify this, we simulate the dynamics with $p = 0.5, q = 0.3$ and choose two initial biases with different growth rates $\Delta(n) = \lceil \log n \rceil$ and $\Delta(n) = \lceil n/10 \rceil$. The results are listed below.

| Number of initial "−" | 50 | 100 | 125 | 150 | 175 | 200 | 225 | 250 |
|---|---|---|---|---|---|---|---|---|
| $\Delta = \lceil \log n \rceil$ | 4 | 5 | 5 | 6 | 6 | 6 | 6 | 6 |
| Winner | "+" | "+" | "+" | "+" | "+" | "+" | "+" | "+" |
| Averaged last day | 5.79 | 16.17 | 35.30 | 70.38 | 183.80 | 445.94 | 1288.29 | 3463.56 |
| Frequency | 982 | 989 | 988 | 998 | 994 | 999 | 991 | 990 |

Table 2: Simulation of the Markovian model when $\Delta = \lceil \log n \rceil, p = 0.5, q = 0.3$.

| Number of initial "−" | 50 | 100 | 125 | 150 | 175 | 200 | 225 | 250 |
|---|---|---|---|---|---|---|---|---|
| $\Delta = \lceil n/10 \rceil$ | 5 | 10 | 13 | 15 | 18 | 20 | 23 | 25 |
| Winner | "+" | "+" | "+" | "+" | "+" | "+" | "+" | "+" |
| Averaged last day | 3.73 | 9.75 | 14.46 | 24.58 | 41.39 | 78.69 | 144.20 | 288.41 |
| Frequency | 1000 | 1000 | 1000 | 1000 | 1000 | 1000 | 1000 | 1000 |

Table 3: Simulation of the Markovian model when $\Delta = \lceil n/10 \rceil, p = 0.5, q = 0.3$.

In Table 2, the frequency of opinion + winning is greater than 98% of all outcomes. Similarly, in Table 3, all simulations end up with opinion + winning. These confirms the theoretical result in Part (ii) of Theorem 2.

Moreover, from the above results, we see that though a greater $\Delta$ reduces the averaged needed time for consensus, the consensus time still exhibits an exponential growth in $n$. This confirms our statement in Remark 4.

## 5.2 Non-Markovian model

For the non-Markovian model, in Theorem 2, we analyzed the phase transition between a fast consensus and halting of the dynamics. We conjectured in Conjecture 1 that this phase transition is sharp and confirmed the conjecture in the special case $p = 1$ in Theorem 3. To verify this

theoretical prediction, we consider a fixed $n = 500$ and set

$$\Delta(n) = \left\lceil \left( \frac{p-q}{q} \right) n - L\sqrt{n \log n} \right\rceil,$$

with varying $L$. In particular, we focus on the outcomes near the critical value

$$L^* = H(p, q) = \frac{\sqrt{p(2-p-q)}}{q}.$$

In our simulations, the last day of the dynamics is averaged over all instances where consensus is reached.

We first focus on the special case $p = 1$ and set $q = 0.3$. In this case, $L^* \approx 2.788$ is the critical value for the phase transition. The simulations are listed below.

| $L$ | $\Delta$ | Winner | Last day | Frequency | $L$ | $\Delta$ | Winner | Last day | Frequency |
|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1167 | "+"/Halt | 2.00 | 1000/0 | 0.5 | 1139 | "+"/Halt | 2.00 | 1000/0 |
| 1.0 | 1110 | "+"/Halt | 2.00 | 1000/0 | 1.5 | 1083 | "+"/Halt | 2.97 | 1000/0 |
| 2.0 | 1055 | "+"/Halt | 3.05 | 1000/0 | 2.5 | 1028 | "+"/Halt | 4.78 | 975/25 |
| 2.7 | 1017 | "+"/Halt | 6.06 | 772/228 | 2.788 | 1012 | "+"/Halt | 7.11 | 541/459 |
| 2.8 | 1011 | "+"/Halt | 7.06 | 505/495 | 3.0 | 1000 | "+"/Halt | 8.69 | 102/898 |
| 3.5 | 972 | "+"/Halt | — | 0/1000 | 4.0 | 944 | "+"/Halt | — | 0/1000 |

**Table 4:** Simulation of the non-Markovian model when $\Delta = \left\lceil \left( \frac{p-q}{q} \right) n - L\sqrt{n \log n} \right\rceil$, $p = 1.0, q = 0.3$.

As shown in Table 4, when $L < L^*$, the opinion $+$ wins with high frequency. In particular, opinion $+$ wins almost surely when $L$ becomes smaller and the consensus becomes faster. On the other hand, when $L > L^*$, the dynamics will halt with high frequency and halting happens almost surely as $L$ gets larger. These observations agree with Theorem 3.

To verify Conjecture 1 in full generality, we now simulate the case where $p = 0.5$ and $q = 0.3$. In this case, the critical value for the phase transition is $L^* \approx 2.582$. The results of the simulation are listed below.

| $L$ | $\Delta$ | Winner | Last day | Frequency | $L$ | $\Delta$ | Winner | Last day | Frequency |
|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 334 | "+"/Halt | 2.00 | 1000/0 | 0.5 | 306 | "+"/Halt | 2.00 | 1000/0 |
| 1.0 | 278 | "+"/Halt | 2.66 | 1000/0 | 1.5 | 250 | "+"/Halt | 3.00 | 1000/0 |
| 2.0 | 222 | "+"/Halt | 4.33 | 1000/0 | 2.5 | 194 | "+"/Halt | 9.50 | 181/819 |
| 2.582 | 190 | "+"/Halt | 9.80 | 66/934 | 2.6 | 189 | "+"/Halt | 10.59 | 54/946 |
| 3.0 | 167 | "+"/Halt | — | 0/1000 | 4.0 | 111 | "+"/Halt | — | 0/1000 |

**Table 5:** Simulation of the non-Markovian model when $\Delta = \left\lceil \left( \frac{p-q}{q} \right) n - L\sqrt{n \log n} \right\rceil$, $p = 0.5, q = 0.3$.

From Table 5, we can see a similar behavior of the outcomes as in the $p = 1$ case. When $L < L^*$, the opinion $+$ wins with high frequency. When $L > L^*$, the dynamics will halt with high frequency. These simulations provide numerical evidence to support the validity of Conjecture 1.

# 6 Concluding Remarks

In this paper, we introduced and analyzed two models for majority dynamics on stochastic block models. For the Markovian model, we showed that any initial bias of the opinions leads to a

uniformly large advantage of the winning probabilities, and gave sufficient conditions for the leading opinion to win a.a.s. For the non-Markovian model, we analyze the phase transition between a fast consensus and a halt of the dynamics. A conjecture is made regarding the sharpness of the phase transition, whose analogue in the case $p = 1$ is confirmed. In addition, in the following we mention a few other interesting directions.

First, as mentioned in the introduction, the $k$-majority dynamics model serves as an alternative to majority dynamics on a static random graph. It is not difficult to formulate $k$-majority dynamics models on SBM with a community structure of those sharing the same opinion. In the literature of $k$-majority dynamics, besides whether consensus is reached, the consensus time is also of great interest. The corresponding analogues require further careful studies.

Second, a model of a similar flavor as our non-Markovian model arises when considering the majority dynamics model for Erdős-Rényi graphs. Pick a constant $K > 1$. Suppose that each agent alters their opinion only if the number of opposite opinions of theirs is at least $K$ times the number of their opinions. Similar questions can be asked such as what difference $\Delta$ guarantees unanimity of the advantageous opinion, and how many days it would take. In this case, the block structure is encoded locally among the agents, instead of globally on the graph.

Third, for technical reasons we have focused on the case where $p, q$ are constants in $(0, 1)$ that do not depend on $n$. In certain voting models, the sparser homogeneous case $p = q \gg n^{-3/5}$ are considered; see [CKLT21, BCO+16]. Extensions of our results in this more general case require further study, where we expect that the recent works of graph enumeration for sparse graphs [LW17, LW20] will be applicable. Note that if $p = a(\log n)/n$, $q = b(\log n)/n$, the two blocks of a stochastic block model are distinguishable only if $|a - b| \geqslant \sqrt{2}$; see [ABH15, MNS14].

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